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X. *On Plane and Spherical Sound-Waves of Finite Amplitude.* *By* CHARLES V. BURTON, *D.Sc.**

PART I.—PLANE WAVES.

1. THE subject of plane waves of finite amplitude has been considered by Riemann †; and so long as we confine our attention to the case where velocity and density are everywhere continuous, his investigation, as is well known, leaves little to be desired. It will not, therefore, be necessary here to make further reference to this aspect of the subject; but there is one part of Riemann's work which Lord Rayleigh has clearly shown to be unsatisfactory, and it is this point which we have now especially to consider. Lord Rayleigh says ‡:—

" . . . . . It has been held that a state of motion is possible in which the fluid is divided into two parts by a surface of discontinuity propagating itself with constant velocity, all the fluid on one side of the surface of discontinuity being in one uniform condition as to density and velocity, and on eht other side in a second uniform condition in the same respects. Now, if this motion were possible, a motion of the same kind

* Read February 24, 1893.

† "Ueber die Fortpflanzung ebener Luftwellen von endlicher Schwingungsweite," *Gött. Abhandl.* t. viii. (1860); reprinted in *Werke*, p. 145.

‡ Theory of Sound, vol. ii. § 253, p. 41.

in which the surface of discontinuity is at rest would also be possible, as we may see by supposing a velocity equal and opposite to that with which the surface of discontinuity at first moves, to be impressed upon the whole mass of fluid. In order to find the relations which must subsist between the velocity and density on the one side (u_1, ρ_1) and the velocity and density on the other side (u_2, ρ_2), we notice in the first place that by the principle of the conservation of matter $\rho_2 u_2 = \rho_1 u_1$. Again, if we consider the momentum of a slice bounded by parallel planes and including the surface of discontinuity, we see that the momentum leaving the slice in the unit of time is for each unit of area $(\rho_2 u_2 = \rho_1 u_1)u_2$, while the momentum entering it is $\rho_1 u_1^2$. The difference of momentum must be balanced by the pressures acting at the boundaries of the slice, so that

$$\rho_1 u_1(u_2 - u_1) = p_1 - p_2 = a^2(\rho_1 - \rho_2),$$

whence

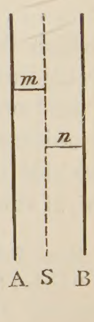
$$u_1 = a\sqrt{\left(\frac{\rho_2}{\rho_1}\right)}, \quad u_2 = a\sqrt{\left(\frac{\rho_1}{\rho_2}\right)}.$$

The motion thus determined is, however, not possible; it satisfies indeed the conditions of mass and momentum, but it violates the condition of energy expressed by the equation

$$\tfrac{1}{2}u_2^2 - \tfrac{1}{2}u_1^2 = a^2 \log \rho_1 - a^2 \log \rho_2."$$

2. The assumed motion here criticised is one in which density and velocity are constant for all points on the same side of the surface of discontinuity, while this surface itself is propagated through the fluid with constant velocity. It is easily shown, however, that the same objection applies when, on either side of the surface, velocity and density vary continuously in the direction of propagation, while the velocity of propagation of the surface is also allowed to vary. For let S (fig. 1) be a surface of discontinuity which is being propagated through the fluid, while the planes A, B, parallel to S and lying on either side of it, are fixed in the fluid. At a given instant let

Fig. 1.



distance of S from A $= m$,

" B " $S = n$;

density and velocity of fluid just to the left of $S = \rho_1, u_1$, density and velocity of fluid just to the right of $S = \rho_2, u_2$; velocity with which S is travelling $= V$.

Then, since A and B are fixed in the fluid, they are approximately moving with the respective velocities u_1, u_2; m and n being taken sufficiently small. On the same understanding, the mass of fluid between A and B (referred to unit surface) $= m\rho_1 + n\rho_2$; and since this mass must remain constant,

$$\frac{d}{dt}(m\rho_1 + n\rho_2) = 0;$$

i. e. in the limit, when m and n are infinitesimal,

$$\rho_1 \frac{dm}{dt} + \rho_2 \frac{dn}{dt} = 0,$$

or

$$\rho_1(V - u_1) = \rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Similarly, if p_1 and p_2 are the pressures corresponding to ρ_1 and ρ_2, the principle of momentum gives:—

$$\begin{aligned} p_1 - p_2 &= \text{rate of change of momentum between A and B} \\ &= \frac{d}{dt}(u_1\rho_1 m + u_2\rho_2 n) \\ &= u_1\rho_1(V - u_1) - u_2\rho_2(V - u_2). \quad . \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

If the energy per unit volume corresponding to density ρ (in the absence of bodily motion) is called $\chi(\rho)$, the principle of energy would further give

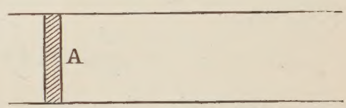
$$\begin{aligned} p_1 u_1 - p_2 u_2 &= \text{rate of change of energy between A and B} \\ &= \frac{d}{dt}\left\{m\left(\tfrac{1}{2}\rho_1 u_1^2 + \chi(\rho_1)\right) + n\left(\tfrac{1}{2}\rho_2 u_2^2 + \chi(\rho_2)\right)\right\} \\ &= \left\{\tfrac{1}{2}\rho_1 u_1^2 + \chi(\rho_1)\right\}(V - u_1) - \left\{\tfrac{1}{2}\rho_2 u_2^2 + \chi(\rho_2)\right\}(V - u_2). \quad (3) \end{aligned}$$

Since (1), (2), and (3) involve only the instantaneous values of u_1, ρ_1, u_2, ρ_2, and V, together with explicit functions of such values, while the space- and time-variations of all these quantities are absent from the equations, it is evident that the conditions to be satisfied at the surface S are the same as if

u_1, ρ_1, u_2, ρ_2, V were absolute constants. We conclude then, that, *with our assumptions*, a surface of discontinuity cannot be propagated through a fluid with any velocity, uniform or variable, except under that special law of pressure for which progressive waves are of accurately permanent type.

3. What, then, becomes of waves of finite amplitude after discontinuity has set in? We may emphasize this difficulty, and at the same time obtain a clue to its solution, by considering the following case (fig. 2):—A is a piston fitting a cylindrical tube (or, if we please, is a portion of an unlimited rigid plane). All the air to the right of A is initially at rest and of uniform density, and then A is impulsively set in motion, and kept moving to the right with uniform velocity v. Consider the speed with which the disturbance generated by A advances into the still air to the right; it is evident that in all cases the front of the disturbance must advance faster than A. Take, then, the case in which

Fig. 2.



$$v > a,$$

where a is the propagation-velocity of infinitesimal disturbances. Two alternatives present themselves :—

(i.) If velocity and density are always either constant or *continuously* variable in the direction of propagation, the rate of propagation at any point will, in accordance with known principles, be

$$\sqrt{\frac{dp}{d\rho}}+u,$$

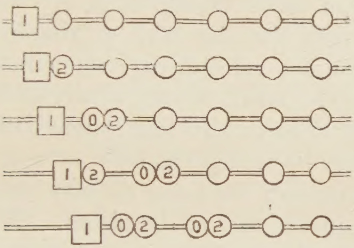
and therefore at the front of the disturbance, where $u=0$ and $\rho =$ the "undisturbed" density, the velocity of propagation will be simply $=a$; that is, *less* than the velocity with which A is advancing. Obviously this will not do.

(ii.) If velocity and density are *not* always either constant or continuously variable, that is, if one or more surfaces of discontinuity are being propagated through the air, we are met by the difficulty explained in the last section.

4. A simple mechanical analogy will help to indicate the actual motion. A number of equal spheres, of the same

material throughout, are capable of sliding without friction along a straight bar (fig. 3), and are connected together by a number of very weak and exactly similar springs (not shown), so that when there is equilibrium they are equally spaced

Fig. 3.



along the bar. If one of the spheres were moved backwards and forwards through a small range, a disturbance would travel through the whole system, but owing to the weakness of the connecting springs it would travel very slowly. Suppose, now, that the last sphere on the left hand is connected to a movable piston by a spring half the length of the others, but otherwise similar to them; and let this piston be suddenly moved to the right with a considerable velocity which is kept constant, and which we may call unity. The weak connecting spring between the piston and the first sphere produces no sensible effect until the two are almost in contact, when the sphere rebounds with velocity 2. This first sphere then strikes the second, imparting to it the velocity 2, and at the same time coming to rest. The positions of the spheres after successive equal intervals of time are represented in fig. 3, where the number written on any sphere represents its velocity just *after* the impact which it is suffering. No number is written on those spheres which have not so far been affected by the motion. From this it will be evident that when the piston moves to the right with a constant velocity which is very great compared with the propagation-velocity of infinitesimal vibrations of the system, the disturbance advances to the right with twice the velocity of the piston, provided that the diameters of the spheres are excluded from the reckoning.

Now suppose that the spheres are too small and too close together to be individually distinguished; then, at any instant,

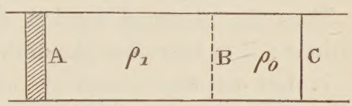
the system will appear to be divisible into two parts, in one of which the velocity is unity, while in the other it is zero; and in the moving part the spheres will appear to be twice as thickly condensed as in the still part. That the constant velocity of the piston is very great compared with the propagation-velocity of small vibrations is of course only a supposition introduced for the sake of simplicity. If, on the other hand, these two velocities are comparable, two adjacent spheres will always remain finitely separated from one another, and the velocity of any individual sphere within the disturbed stretch will never be as small as zero, or as great as twice the velocity of the piston; the *mean* velocity within the disturbed stretch being equal to that of the piston. When the spheres are very small and very close together, we shall still have apparently an abrupt transition from finite velocity and greater density to zero velocity and smaller density; and the energy, which is apparently lost as the spheres pass from the latter condition to the former, exists as energy of relative motion and unequal relative displacement amongst the spheres in the disturbed stretch.

5. Let us now compare the case just considered with the case of § 3 (fig. 2): and first, concerning the nature of the analogy, it should be noticed that the individual spheres are not the analogues of the separate gaseous molecules, but that when both spheres and molecules are very small and very numerous, the apparently continuous properties of the system of spheres correspond to similar properties of the gas. The connecting springs represent the elasticity of the gas, isothermal or adiabatic as the case may be, and the energy of relative motion and unequal relative displacement amongst the disturbed spheres *suggests* that there is a production of heat over and above that which would be due to the (isothermal or adiabatic) change of density; that is, *a dissipative production of heat*. The motion considered in the last section properly corresponds to the case where there is no conduction of heat, so that the connecting springs are the representatives of *adiabatic* elasticity, and the additional heat generated remains wholly within the more condensed part of the air. If we make the somewhat violent assumption that the temperature of the air remains constant throughout, the additional

heat generated will be conducted away isothermally, and the equivalent energy will be, for our purposes, entirely lost. To represent this case by means of our spheres we should have to regard the connecting springs as representing isothermal elasticity, while the energy of relative motion and unequal relative displacement among the disturbed spheres, as fast as it is produced, is to be consumed in doing work against suitable internal forces.

6. The mechanical system of spheres and springs, having suggested a solution, has served its purpose, and it now remains for us more closely to consider the aerial problem in the light of this suggestion. We may take, first, the case where the temperature is supposed to be invariable; for although such a supposition is necessarily far removed from the truth, it leads to very simple results, which indicate well enough the general character of the motion. Let the piston A (fig. 4) be moving to the right with constant velocity v (which may be either less or greater than a, the velocity of feeble sounds in air). Assume all the air between A and a parallel plane surface B to have the velocity v and density ρ_1, while all the air to the right of B is at rest and has the density ρ_0. Let the plane B move to the right with velocity V. Then the invariability of mass between A and a plane C fixed in the still air gives

Fig. 4.



$$\rho_1(\mathrm{V}-v)-\rho_0 v=0\,; \qquad (4)$$

while from the principle of momentum,

$$\rho_1 v(\mathrm{V}-v)=p_1-p_0\,; \qquad (5)$$

the pressure p being a function of ρ only, since the temperature is supposed to be constant throughout. If we assume for this case the truth of Boyle's law, so that $p=a^2\rho$ always, (5) becomes

$$\rho_1(a^2-\mathrm{V}v+v^2)=\rho_0 a^2, \qquad (6)$$

which together with (4) is sufficient to determine V and ρ_1 when v and ρ_0 are given. Taking all these quantities to

remain constant throughout the motion, we see that at each instant the following conditions are satisfied :—

(i) Every necessary condition between A and B, since density and velocity are there constant with respect to space and time ;

(ii) Every necessary condition to the right of B, since the air there is at rest and in a constant uniform state ;

(iii) Equality between the velocity of A and that of the air in contact with it ;

(iv) At B, the conservation of mass and momentum, which are *necessary* conditions, and which, together with our supposition that the temperature is somehow maintained uniform, are *sufficient* to determine what takes place at B *.

Moreover, if at a time t (reckoned from the instant when A was impulsively started into motion) we take the distance of B from A to be $(V-v)t$, so that initially B coincides with A, the initial conditions are satisfied.

Thus the assumed motion satisfies all the necessary conditions ; it is therefore the actual motion.

7. Let us now examine what occurs when no heat is allowed to pass by conduction or radiation ; a state of things much more nearly realized in practice. Suppose the motion of A and the condition of the undisturbed air to be the same as in the last section, while the (constant) velocity of B is now called V', and the density and pressure of the air between A and B (called ρ', p' respectively) are also taken to be uniform and constant. At each instant, in place of (4) and (5), we shall now have

$$\rho'(V'-v)-\rho_0 v=0, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$\rho' v(V'-v)=p'-p_0. \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Since we assume that there is no transference of heat by conduction or radiation, the rate at which the *total* energy of the system increases must be equal to the rate at which work is being done upon it by the piston A. Let θ_0 be the abso-

* Energy appears to be lost, because dissipatively produced heat is conducted away isothermally.

lute temperature to the right of B, that between A and B being θ', and let us further assume for simplicity that

$$\frac{p}{\rho\theta}=\text{a const.};$$

while γ, the ratio of the two specific heats, is also supposed constant. It can then be shown without difficulty that the total energy *per unit mass* between A and B exceeds that to the right of B by

$$\frac{p_0}{(\gamma-1)\rho_0}\,\frac{\theta'-\theta_0}{\theta_0}+\frac{v^2}{2};$$

and multiplying this by $\rho_0 V'$, the mass of air which crosses one unit of the surface B in each unit of time, we obtain the rate (referred to unit area) at which the system is gaining energy. Again, the rate at which unit area of the piston does work on the system

$$=p'v=p_0\frac{\rho'\theta'}{\rho_0\theta_0}\cdot v,$$

and equating this to the rate of gain of energy, we obtain

$$p_0\frac{\rho'\theta'}{\rho_0\theta_0}\,v=\rho_0 V'\frac{v^2}{2}+\frac{p_0 V'}{(\gamma-1)\theta_0}(\theta'-\theta_0). \quad . \quad . \quad (9)$$

We may also write equation (8) in the form

$$\rho' v(V'-v)=\frac{p_0}{\rho_0\theta_0}(\rho'\theta'-\rho_0\theta_0); \quad . \quad . \quad . \quad (10)$$

and (7), (9), and (10) will then serve to determine V', ρ', θ' when v, ρ_0, θ_0 are given. Since we have taken all these quantities to remain constant throughout the motion, we see, as before, that at each instant all the necessary conditions are satisfied; the principles of mass and momentum, together with our supposition that there is no exchange of heat, being sufficient to determine what takes place at B. Again, if at a time t from the commencement of the motion we take the distance of B from A to be $(V'-v)t$, so that initially B coincides with A, the initial conditions are satisfied. The assumed motion thus satisfies all the necessary conditions, and is therefore the actual motion.

8. If we compare the results of the last two sections with

those given by Riemann*, we shall find complete accordance so far as § 6 is concerned, though with § 7 the case is different; and this may be easily explained. We cannot in general investigate the motion of a (frictionless) compressible fluid by means of the equations of continuity and momentum, without further making some supposition as to the exchange or non-exchange of heat, and so we usually assume either that the temperature remains constant, or that there is no exchange of heat: in either case (provided the motion is continuous), the pressure is a function of the density only. At a surface of discontinuity there is not only the ordinary heating effect due to compression, but also, as we have seen, a dissipative generation of heat, and so, when applying the equations of continuity and momentum at such a surface, we must know what becomes of this additional heat. Now in all cases Riemann makes the assumption that the pressure is a function of the density only, and this is necessarily equivalent to an assumption concerning the transference of heat. Throughout most of his treatment of waves of discontinuity Riemann assumes that temperature is constant and that Boyle's law holds good; accordingly our § 6 is entirely in harmony with his conclusions, in fact (4) and (5) are only particular forms of equations given by Riemann. Of course the hypothesis that a portion of gas can be instantaneously compressed to a finite extent without any appreciable change of temperature, is not in accordance with experience, *but provided we accept the assumption that the temperature remains constant throughout, all that Riemann says concerning the propagation of waves of discontinuity under Boyle's law will hold good.*

The assumption made in § 7, that there is no appreciable transference of heat, is probably much nearer the truth; but this is not in accordance with any assumption made by Riemann. When pressure is assumed to be a function of density only, and to vary with it according to the adiabatic law, *it is virtually assumed that at the discontinuity just so much heat remains in the gas as would be due to slow adiabatic compression, while the further amount of heat which is dissipa-*

* *Loc. cit.*

tively produced is completely and instantaneously removed by conduction. But though Riemann's results may thus be justified by impossible assumptions concerning the diffusion of heat, we may more reasonably, following Lord Rayleigh, regard them as involving a destruction of energy. The real source of error lies in Riemann's fundamental hypothesis. At the outset he supposes the expansion and contraction of the air to be either purely isothermal or purely adiabatic, and thenceforward he treats the air as a frictionless and mathematically continuous fluid, in which pressure and density are connected by an invariable law. But in general the existence of such a fluid is contrary to the conservation of energy; for as soon as discontinuity arises, energy will be destroyed.

9. It may not be out of place to conclude this portion of the subject by a short reference to a paper by Dr. O. Tumlirz *. This author starts, as Riemann did, with the assumption that the pressure is a function of the density only, the law of pressure being further assumed to be the adiabatic law; and in order to avoid Riemann's error, he explicitly uses the principle of energy applicable to continuous motion, in place of the principle of momentum. But the foregoing discussion will have made it clear, I think, that the solution of the difficulty is not to be sought for in this direction. In addition to the assumptions common to his own work and to that of Tumlirz, Riemann uses only the principle of mass and the principle of momentum; and since by their aid alone he arrives at a completely determinate motion, it follows that any other motion consistent with the same arbitrary assumptions, and with the condition of mass, must violate the condition of momentum. We have seen, in fact, that there is dissipation of energy at a surface of discontinuity, so that the condition of energy applicable to continuous motion ceases to hold good. We are acquainted, too, with other instances where loss of continuity involves dissipation of energy; for example, there is the case of one hard body rolling over another.

As the result of his investigation, Dr. Tumlirz concludes

* "Ueber die Fortpflanzung ebener Luftwellen endlicher Schwingungsweite," *Sitzungsb. der Wien. Akad.* xcv. pp. 367–387 (1887).

that as soon as a discontinuity is formed it immediately disappears again, this effect being accompanied by a lengthening of the wave and a more rapid advance of the disturbance. In this way, therefore, he seeks to explain the increased velocity of very intense sounds, such as the sounds of electric sparks investigated by Mach *. But it has already been pointed out [§ 3 (i.)], that *when density and velocity are everywhere continuous functions of the coordinates*, the *front* of a disturbance advancing into still air must travel forward with the velocity of infinitely feeble sounds. A greater velocity can only ensue when the motion has become discontinuous.

Part II.—Spherical Waves.

10. When plane waves of finite amplitude are propagated through a frictionless compressible fluid, discontinuity must always occur sooner or later, and a moment's consideration will show that there are at least some cases when the motion in spherical waves becomes discontinuous; the question arises whether in any case it is possible (in the absence of viscosity) for divergent spherical waves to travel outward indefinitely without arriving at a discontinuous state. This question was suggested to me by Mr. Bryan, who at the same time kindly handed me notes of his manner of attacking the problem. His method was to write down the exact kinematical equation for spherical sound-waves, and then to obtain successive approximations to the integral of this equation. If it appears that after any number of approximations the integral would remain convergent for large values of the radius, we may conclude that our equation holds good throughout, and hence that no discontinuity arises. If, on the other hand, the second or any higher approximation becomes divergent for large values of the radius, it is probable that the motion becomes somewhere discontinuous. This method I have not followed out; but by another method which is, I hope, sufficiently conclusive, I shall now endeavour to show that discontinuity must always arise.

* *Sitzungsb. der Wien. Akad.* lxxv., lxxvii., lxxviii. *Cf.* also W. W. Jacques [On Sounds of Cannon], Amer. Journ. Sci. 3rd ser. xvii. p. 116 (1879).

The case in which the motion loses its continuity comparatively early requires no further consideration here ; we have only to concern ourselves with the case in which the initial disturbance has spread out into a spherical shell of very small disturbance whose mean radius is very great compared with the difference between its extreme radii. The equations applicable to the disturbance are then, very approximately,

$$u=a\frac{\delta\rho}{\rho}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$u \text{ or } a\frac{\delta\rho}{\rho} \propto \frac{1}{r} \text{ for a given part of the wave}, \quad . \quad (12)$$

where ρ is the mean density, $\rho+\delta\rho$ the actual density at a point where the velocity is u, and a is the velocity of infinitely feeble sounds in air of density ρ ; r is as usual the distance of a point from the centre of symmetry. Let us consider two neighbouring points M and N, on the same radius, *each being fixed in a definite part of the wave*, the point M being *behind* N (*i. e.* nearer to the origin), and the air-velocity at M exceeding that at N by Δu. Then, as the wave advances, each part of it will be instantaneously moving forward with (very approximately) the velocity

$$\sqrt{\frac{dp}{d\rho}}+u,$$

determined by the corresponding values of ρ and u ; so that M will be gaining on N at the rate

$$\Delta u+\frac{d}{d\rho}\sqrt{\frac{dp}{d\rho}}\cdot\frac{d\rho}{du}\Delta u \quad . \quad . \quad . \quad . \quad . \quad (13)$$

approximately. We may admit then that the rate at which M gains on N is

$$\text{never} < \mathrm{B}\Delta u,$$

where B is a constant suitably chosen.

Again, if $\Delta_0 u$ is the difference between the air-velocities at M and N at the time $t=0$, and r_0 is the corresponding coordinate of M, we may admit that

$$\Delta u \text{ is never } < \frac{\mathrm{A}r_0}{r_0+at}\Delta_0 u,$$

where A is a constant not very different from unity. Thus M gains on N at a rate which is

$$\text{never} < \mathrm{AB}\frac{r_0}{r_0+at}\Delta_0 u\,;$$

and between the times $t=0$ and $t=t_1$ the distance gained by M relatively to N will be

$$\text{at least } \mathrm{AB}\Delta_0 u\int_0^{t_1}\frac{r_0\,dt}{r_0+at},$$

$$i.\,e. \qquad \text{at least } \mathrm{AB}\Delta_0 u\frac{r_0}{a}\log\frac{r_0+at_1}{r_0}. \quad . \quad . \quad . \quad (14)$$

If B is finite and positive this expression increases indefinitely with the time, so long as the laws of continuous motion hold good. If $\Delta_0 r$ was the distance between M and N at time $t=0$, the time required for M to *overtake** N will be *not greater* than the value of t_1 given by

$$-\Delta_0 r = \mathrm{AB}\Delta_0 u\frac{r_0}{a}\log\frac{r_0+at_1}{r_0}\,;$$

or, when M and N are taken indefinitely close together at starting, by

$$\log\frac{r_0+at_1}{r_0} = \frac{a}{r_0} \div \left\{\mathrm{AB}\left(-\frac{\partial u}{\partial r}\right)_0\right\};$$

$$i.\,e.,\text{ we have } t_1 \leqq \frac{r_0}{a}\left(e^{\frac{a}{r_0}\div\left\{\mathrm{AB}\left(-\frac{\partial u}{\partial r}\right)_0\right\}} - 1\right), \quad . \quad . \quad (15)$$

which gives us a finite upper limit to the time required for discontinuity to set in, provided B is finite. As our assumptions only remain approximately true so long as the motion is continuous, (15) will only give an approximation to the time when discontinuity first commences, and accordingly the relation must be taken to refer to that part of the wave for which its right-hand side is a minimum. If B is negative (which is not the case for any known substance), the appropriate part of the disturbance will be such that $\partial u/\partial r$ is positive.

To determine approximately the value of B, we may refer to (13) and the inequality immediately following. If we

* *Cf.* Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 36.

assume Boyle's law of pressure, so that $\surd(dp/d\rho)=\text{const.}$, we have evidently

$$B=1 \text{ very nearly.}$$

If we assume that the changes of density take place adiabatically, so that $p \propto \rho^\gamma$ and γ is nearly constant, the approximate value of B becomes

$$1+\frac{d}{d\rho}\sqrt{\frac{dp}{d\rho}}\cdot\rho\div\sqrt{\frac{dp}{d\rho}}$$

by means of (11);

$$=\frac{\gamma+1}{2}.$$

If, then, viscosity be neglected, we must conclude that under any practically possible law of pressure the motion in spherical sound-waves always becomes discontinuous, and *à fortiori* the same will be true of cylindrical waves. But inasmuch as our result for spherical waves depends on the existence of an infinite *logarithm* in (14) when t_1 is increased without limit, we may conclude that for waves diverging in four dimensions (or, more generally, in any number of dimensions finitely greater than three) there would be some cases where the motion remained always continuous.

11. The general question of spherical sound-waves of finite amplitude is by no means an easy one. In the case of plane waves we can write down at once from Riemann's equations the condition that the disturbance may be propagated wholly in the positive or wholly in the negative direction. The respective conditions are * :—

$$u=\pm\int_{\rho_0}^{\rho}\sqrt{\frac{dp}{d\rho}}\cdot d\log\rho,$$

where ρ_0 is the density of that part of the fluid whose velocity is reckoned as zero. No such simple criterion can be given for the existence of a purely convergent or purely divergent spherical disturbance; a fact which may be readily seen from the equations for waves of infinitesimal amplitude. If ϕ is the potential of a purely divergent system of waves, we have

$$r\phi=f(at-r), \quad \ldots \ldots \quad (16)$$

* *Cf.* also Lord Rayleigh, 'Theory of Sound,' vol. ii. p. 35 (3).

where f is a function whose form is unrestricted. Let ρ be the ordinary density of the air, and $\rho+\delta\rho$ the actual density at a point where the coordinate is r and velocity u. We have, then, on differentiating (16) the well-known relations

$$u=\frac{\partial\phi}{\partial r}=-\frac{f(at-r)}{r^2}-\frac{f'(at-r)}{r} \quad . \quad . \quad . \quad (17)$$

and

$$a\frac{\delta\rho}{\rho}=-\frac{1}{a}\frac{\partial\phi}{\partial t}=-\frac{f'(at-r)}{r}. \quad . \quad . \quad . \quad . \quad (18)$$

From (17) and (18),

$$\left(a\frac{\delta\rho}{\rho}-u\right)r^2=f(at-r),$$

whence differentiating with respect to r, and neglecting small quantities beyond the first order,

$$r^2\left(\frac{a}{\rho}\frac{\partial\rho}{\partial r}-\frac{\partial u}{\partial r}\right)+2r\left(a\frac{\delta\rho}{\rho}-u\right)=-f'(at-r)$$
$$=ar\frac{\delta\rho}{\rho}$$

by (18); therefore

$$\frac{a}{\rho}\frac{\partial\rho}{\partial r}-\frac{2u}{r}+\frac{a}{r}\frac{\delta\rho}{\rho}-\frac{\partial u}{\partial r}=0. \quad . \quad . \quad . \quad . \quad (19)$$

If, then, an infinitesimal spherical disturbance is to be purely divergent, this equation must be satisfied for every value of r. But since the left-hand side involves $\delta\rho/\rho$ as well as u, $\partial u/\partial r$, and $\partial(\log\rho)/\partial r$, it is evident that the question whether or not the equation is satisfied for some particular value of r does not depend solely on the state of things in the immediate neighbourhood of this value, but is influenced also by the value of ρ corresponding to the undisturbed air. We must not therefore seek to characterize a purely divergent disturbance by a differential equation expressing that, with respect to the air at each point, the disturbance is wholly propagated in the positive direction of r.

12. Not recognizing this, I had attempted to discover such an equation, and one step of the inquiry is reproduced here, for the sake of any interest which it may have.

It is required to write down the differential equation of an

infinitesimal spherical disturbance, which is superposed on a purely radial steady motion.

Though a steady motion extending inward to the pole would involve a violation of the principle of continuity, we may suppose that throughout a shell of finite thickness the distribution of density and velocity is such as would be consistent with steady motion ; the motion within such a shell would then continue steady, provided that its spherical boundaries were constrained to expand or contract in a suitable manner. In the absence of constraints the shell of steady motion would be invaded from without and from within by disturbances emanating from adjoining parts of the fluid, but, at points well within the shell, the character of the steady motion would necessarily be maintained for a finite time.

Let ϕ be the potential of the steady motion.

Let $\phi+\psi$ be the potential of the actual motion so that ψ and its derivatives are small.

Let p, ρ be the pressure and density in the steady motion.

Let $p+\delta p$, $\rho+\delta\rho$ be the pressure and density in the actual motion, and assume that the pressure is a function of the density only. From the ordinary equations for the motion of compressible fluids we obtain

$$\int^{p}\frac{dp}{\rho}=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^{2}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

$$\int^{p+\delta p}\frac{dp}{\rho}=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}+\frac{\partial\psi}{\partial r}\right)^{2}-\dot{\psi}$$

$$=-\tfrac{1}{2}\left(\frac{\partial\phi}{\partial r}\right)^{2}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}-\dot{\psi}, \quad . \quad . \quad . \quad (21)$$

when small quantities of the second order are neglected. Subtracting (20) from (21),

$$\frac{\delta p}{\rho}=-\dot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}. \quad . \quad . \quad . \quad . \quad . \quad (22)$$

Now

$$\frac{\delta p}{\rho}=\frac{dp}{d\rho}\cdot\frac{\delta\rho}{\rho};$$

therefore

$$\frac{\partial}{\partial t}\frac{\delta p}{\rho}=\frac{dp}{d\rho}\cdot\frac{1}{\rho}\frac{\partial \cdot \delta\rho}{\partial t}; \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and the equations of continuity for the steady motion and the actual motion may be written

$$0=\frac{\partial \rho}{\partial t}=-\rho\nabla^2\phi-\frac{\partial\phi}{\partial r}\frac{\partial\rho}{\partial r},$$

$$\frac{\partial(\rho+\delta\rho)}{\partial t}=-(\rho+\delta\rho)\nabla^2(\phi+\psi)-\left(\frac{\partial\phi}{\partial r}+\frac{\partial\psi}{\partial r}\right)\frac{\partial(\rho+\delta\rho)}{\partial r},$$

whence by subtraction

$$\frac{\partial\delta\rho}{\partial t}=-\rho\nabla^2\psi-\delta\rho\nabla^2\phi-\frac{\partial\rho}{\partial r}\frac{\partial\psi}{\partial r}-\frac{\partial\delta\rho}{\partial r}\frac{\partial\phi}{\partial r}. \quad . \quad (24)$$

Again,

$$\frac{\partial\delta\rho}{\partial r}=\frac{\partial}{\partial r}\left\{\left(\frac{dp}{d\rho}\right)^{-1}\delta p\right\}$$

$$=\frac{\partial}{\partial r}\left\{\rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\dot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\right\};$$

expanding this and substituting in (24) we get

$$\frac{\partial\delta\rho}{\partial t}=-\rho\nabla^2\psi-\rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\dot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\nabla^2\phi-\frac{\partial\rho}{\partial r}\frac{\partial\psi}{\partial r}$$

$$-\left[\frac{\partial}{\partial r}\left\{\rho\left(\frac{dp}{d\rho}\right)^{-1}\right\}\left(-\dot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\psi}{\partial r}\right)\right.$$

$$\left.+\rho\left(\frac{dp}{d\rho}\right)^{-1}\left(-\frac{\partial\dot{\psi}}{\partial r}-\frac{\partial\phi}{\partial r}\frac{\partial^2\psi}{\partial r^2}-\frac{\partial^2\phi}{\partial r^2}\frac{\partial\psi}{\partial r}\right)\right]\frac{\partial\phi}{\partial r} \quad . \quad (25)$$

Now differentiate (22) with respect to t and we have, remembering (23),

$$-\ddot{\psi}-\frac{\partial\phi}{\partial r}\frac{\partial\dot{\psi}}{\partial r}=\frac{1}{\rho}\frac{dp}{d\rho}\cdot\frac{\partial\delta\rho}{\partial t}.$$

In this equation we have to substitute the value of $\partial\delta\rho/\partial t$ from (25), and if we then put $\chi\equiv\psi r$, and perform the necessary reductions, we finally obtain as the differential equation satisfied by ψr,

$$\frac{\partial^2\chi}{\partial t^2}-(a^2-u^2)\frac{\partial^2\chi}{\partial r^2}+2u\frac{\partial^2\chi}{\partial r\partial t}+\mathrm{U}\frac{\partial\chi}{\partial t}+\mathrm{V}\left(\frac{\partial\chi}{\partial r}-\frac{\chi}{r}\right)=0,$$

where

$$a^2 = \text{the variable } \frac{dp}{d\rho} \text{ (in the steady motion)},$$

$$u = \frac{\partial\phi}{\partial r} = \frac{\text{constant}}{\rho r^2},$$

$$U = \frac{1}{\rho}\frac{\partial(u\rho)}{\partial r} - \frac{2u}{a}\frac{\partial a}{\partial r} = -\frac{2u}{r} - \frac{2u}{a}\cdot\frac{\partial a}{\partial r},$$

$$V = 2au\frac{\partial}{\partial r}\left(\frac{u}{a}\right) - \frac{a^2-u^2}{\rho}\frac{\partial\rho}{\partial r}.$$

If the steady motion in question is a state of rest, $u=0$ and ρ is a constant, so that $U=0$, $V=0$, and our equation reduces to the ordinary form for small spherical disturbances,

$$\frac{\partial^2\chi}{\partial t^2} - a^2\frac{\partial^2\chi}{\partial r^2} = 0.$$

If, on the other hand, $r=\infty$, the motion may, through any finite distance, be treated as linear. We shall then have u and ρ both constant, as well as a, and as before $U=0$, $V=0$. In that case

$$\frac{\partial^2\psi}{\partial t^2} - (a^2-u^2)\frac{\partial^2\psi}{\partial r^2} + 2u\frac{\partial^2\psi}{\partial r\partial t} = 0,$$

and this, by a change of independent variables, is easily seen to be the appropriate form for small plane disturbances of a fluid whose motion is otherwise uniform.

Discussion.

Prof. A. S. Herschel enquired whether the nature of the solution for plane waves of finite amplitude was similar to that for ordinary wave-motion? In the latter case everything depended on the instantaneous impulses, for these alone determined the nature of the wave.

Referring to Mr. Bryan's paper, he (Prof. Herschel) asked if the author could apply his equations to centrifugal fans. A particular kind of double fan had recently been tested, and gave very anomalous results which had not yet been explained.

The President said Mr. Boys's experiments on flying-bullets might have some bearing on Dr. Burton's paper. If the conclusions there stated were correct, then the velocity of the air in front of a bullet should be greater than that of the bullet, even if the latter was travelling faster than ordinary

sound-waves. He now asked Mr. Boys if his photographs gave any evidence of this.

Mr. Boys said the fact that the photographs showed disturbances in front of the bullet proved that the disturbance travelled faster. In one case, where a large bullet was moving at a velocity rather greater than that of ordinary sound in the medium, the front of the disturbance was about half an inch in advance of the bullet. In another instance, where the bullet was smaller and the velocity greater, the distance which the disturbance was in advance of the bullet was somewhat less. In all cases, even when the velocity of the bullet was four times that of sound, the character of the effects remained the same.

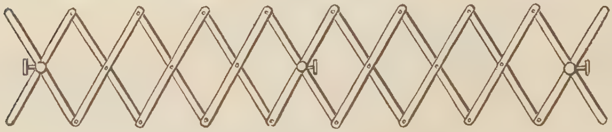
Dr. Burton replied to the points raised.

XI. *On a new and handy Focometer.* *By* Professor J. D. Everett, *F.R.S.**

This focometer is designed to permit the distance of the "object" from the screen to be varied, while the lens which is to throw on the screen an image of the "object" is automatically kept midway between the two. This position, as is well known, gives both the sharpest definition and the simplest calculation.

The instrument is constructed on the principle of the well-known toy called *lazy-tongs*. A number of flat bars (fig. 1), all exactly alike, are jointed together in such a way that half

Fig. 1.



Plan.

of them are in one plane and the other half in a superposed plane. With the exception of the end bars, each bar in either plane is jointed to three of the bars in the other plane, one

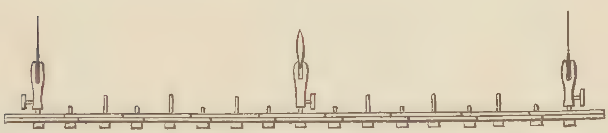
* Read February 24, 1893.

joint being in the middle and one at each end. The end bars are jointed at the middle and one end only. All the bars in the same plane are parallel, and the two sets together form a single row of rhombuses all equal and similar, a side of a rhombus being half the length of a bar. The system has only one degree of freedom, and its length is a definite multiple of the longitudinal diagonal of a rhombus.

The joints are arranged in three rows, one down the middle and one along each edge, and the distance from joint to joint in any row is equal to this longitudinal diagonal. This common distance can be varied between very wide limits by pulling out or pushing in the frame, and we have thus a means of dividing an arbitrary length into any number of equal parts. I utilize only the middle row for this purpose, and utilize it only or chiefly for bisecting a variable distance.

The pins on which the middle joints turn are continued

Fig. 2.

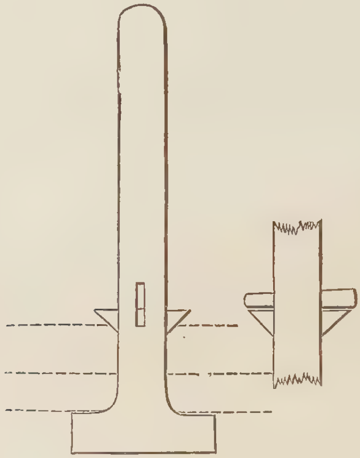


Elevation.

upwards, as shown in fig. 2, to serve as supports for clips holding the object, the lens, and the screen. The lens is mounted on the centre pin, and the object and screen usually on the two end pins, as in fig. 2. In order to avoid flexibility, the clips are made short, and the pins, on which they are held by screws, rise only $1\frac{1}{2}$ inch above the frame. The base of each pin is a substantial disk (see fig. 3) which rests upon the table ; all the pins, not only in the middle row, but also in the two outside rows, terminate in such disks, which serve as the feet of the instrument, and slide upon the table when the frame is expanded or contracted. The pins are of brass $\frac{1}{4}$ inch in diameter, and the bars are of $\frac{1}{4}$-inch mahogany, $\frac{3}{4}$ inch wide, and 13 inches in gross length. There are 18 of them, as shown in fig. 1. There are 9 pins in the middle row ; and when the object and screen are on the two end pins, the distance between them is divided by the other pins into 8 equal parts, any two of

which should together make up the focal length. The unused pins are the most convenient handles for manipulating the frame.

Fig. 3.



A supporting pin.

The screen may conveniently be a piece of white card a little larger than a post-card, and a square of wire-gauze about half as big may be used as the object; but a still better 'object' is a cross of threads stretched across a square hole in a card. The light which passes through the square hole is very conspicuous on the screen before the correct distance is approached, whereas the shadow of the wire gauze is almost invisible. Two thin cards about the size of post-cards should be taken, a hole a centimetre square should be cut through both of them, and they should be gummed together with the cross threads between them, the threads being in the first instance long enough to project beyond the cards to facilitate adjustment while the gum is wet. Waxed carpet-thread, or any very stout thread with smooth edges, is the best for giving a conspicuous and at the same time a sharp image. As the cross will sometimes have to be raised or lowered, the hole should be much nearer to one end of the card than to the other, in order to give a greater range of adjustment in mounting on the clip. One thread should be vertical and the other horizontal, in order that their simul-

taneous focussing may serve as a check on the correct orientation of the lens.

The instrument is intended to be used by placing it on a table of length not less than four times the focal distance which is to be measured. A lamp is to be placed either on one end of the table or on a stand opposite the end, at such a height that its flame is about level with the tops of the clips. The clips should be fixed as low as possible on their supporting pins, unless it is necessary to raise them to suit the height of the lamp. In default of a lamp at the proper height, an adjustable mirror may be used instead, and made to reflect a beam of light from any large gas-flame in the room so that the beam shall pass along the tops of the clips. When the lamp or mirror has once been adjusted to throw its light in the proper direction, it should not be disturbed, as all necessary adjustments can be better made by moving the instrument.

The lens and screen may conveniently be mounted first, and the adjustments made so that the light collected by the lens falls on the screen as a horizontal beam. The cross is then to be mounted in such a position that a bright patch corresponding to the square hole is seen on the screen, surrounded by the shadow of the card. The frame must now be extended or compressed till the image of the cross appears in the bright patch; and the lens, object, and screen should then be carefully set square by hand before the final adjustment. If the vertical and horizontal lines of the cross do not focus simultaneously, it is a sign that the lens needs setting square.

The focussing having been completed, the distance of the object from the image is to be measured and divided by four. This will give the focal length; and the calculation can be checked by measuring one or more of the four equal parts into which the distance is divided by alternate pins. Owing to slight play in some of the joints, or other mechanical imperfections, the theoretically equal distances may exhibit sensible differences, especially when the frame is nearly closed up; but the method of observation is so well conditioned that these inequalities do not practically affect the correctness of the result.

In fact, if the distances of the lens from the object and

image, instead of being exactly equal, are $a+x$ and $a-x$, the true focal length is $\frac{a^2-x^2}{2a}$, and in taking it to be one fourth of the whole distance we are simply neglecting x^2 in comparison with a^2. Suppose the two distances $a+x$ and $a-x$ to measure $20\frac{1}{2}$ and $19\frac{1}{2}$ inches, which is a larger inequality than is likely to occur, the ratio of x^2 to a^2 is 1 to 1600 ; and this error is negligible, in view of the fact that the doubt as to when the image is sharpest involves an uncertainty in the focal length to the extent usually of more than one per cent.

When the focal length does not exceed 10 or 12 inches, the instrument may be supported with the two hands and pointed towards a gas-flame, which need not be at the same level, but may be at any height. A fairly good measurement can thus be made by one person, if there is opportunity for setting the instrument down on a table or floor when the lens needs setting square, and when the final measurement of distance is to be made. The friction at the joints of the frame is just sufficient to keep them from working while the instrument is being carefully set down. The chief difficulty is from flexure.

Instead of receiving the image on a screen, it can be viewed in mid air. For this purpose I mount the cross on one of the two end clips, and a piece of wire gauze about the size of the palm of my hand on the other, setting the wires at a slope of 45° by way of contrast with the upright cross. The end which carries the cross should be turned towards the strongest light ; as this renders the cross more visible to an observer behind the gauze, and also renders the glistening wires of the gauze more visible when the observer stations himself behind the cross. The adjustment for focus is made by lengthening or shortening the frame till parallax is removed. This is a very convenient way of establishing experimentally the fact of the interchangeableness of object and image.

The instrument can also be employed to illustrate the general law of variation of conjugate focal distances, the lens being for this purpose shifted from the central pin to any one of the other pins, and the frame being then extended till the image is correctly focussed. Regarded as an optical bench,

the instrument is remarkably light and handy. Its weight, including screen, cross, wire-gauze, and lens, is 2 lb. 10 oz.; and a lecturer can carry it through the streets of a town without inconvenience.

The dimensions and number of bars of the instrument as exhibited are recommended as the most convenient for general purposes. Ten bars only were constructed for the first trials, and any number included in the formula $4n+2$ might theoretically be employed.

In order to prevent looseness at the joints, it would be well to make the holes in the bars bear against a cone below and another cone above, with a very slightly tapering wedge for adjustment, as indicated in fig. 3.

If the instrument were to be set up permanently in one place, guides might be used for compelling the middle row of pins to travel without rotation, or the pin on which the lens is mounted might be a fixture; but as long as portability is to be preserved, I do not think that any arrangements for automatically preventing rotation would be practically beneficial. It is only in the large movements which precede the final adjustment that rotation occurs to any injurious extent.

The instrument has been constructed from my drawings by Messrs. Yeates of Dublin, and the cost is trifling.

Discussion.

Mr. A. Hilger thought the instrument was too flexible to be used for accurate work.

Mr. Blakesley said it would be a great improvement if the pins could be prevented from rotating. For this purpose it might be advantageous to slot the heads of the pins so as to fit on a straight bar. He also suggested that by using a plane mirror close behind the lens the light would be reflected back and the length of the focometer could be reduced by one half.

The President thought Prof. Everett never intended the instrument to compete, as regards accuracy, with the elaborate and expensive apparatus now used; but nevertheless the focometer was a very valuable one, especially for students' work, and was particularly well adapted to impress upon them the facts relating to conjugate foci.

XII. *A Hydrodynamical Proof of the Equations of Motion of a Perforated Solid, with Applications to the Motion of a Fine Rigid Framework in Circulating Liquid.* By G. H. Bryan*.

Introduction.

1. In the whole range of hydrodynamics, there is probably no investigation which presents so many difficulties as that which deals with the equations of motion of a perforated solid in liquid. The object of the present paper is to show how these equations may be deduced directly from the pressure-equation of hydrodynamics, without having recourse to the laborious method of ignoration of coordinates. The possibility of doing this is mentioned by Prof. Lamb in his 'Treatise on the Motion of Fluids' (pp. 119, 120), but he dismisses the method with the brief remark that in most cases it would prove exceedingly tedious. I think, however, that it will be admitted that the following investigation is more straightforward and simple than that given by Basset in his 'Hydrodynamics,' vol. i. pp. 167–178.

The usual method presents little difficulty when the motion of the liquid is acyclic, because the whole motion could in such cases be set up from rest by suitable impulses applied to the solids alone; and a consideration of Routh's modified Lagrangian function shows that in this case the equations of motion can be obtained by expressing the total kinetic energy as a quadratic function of the velocity-components of the solid alone, and applying the generalized equations of motion referred to moving axes.

If, however, the solid is perforated, and the liquid is circulating through the perforations, this method presents several difficulties. If the solid were reduced to rest by the application of suitable impulses, the liquid would still continue to circulate through the perforations, the "circulation" in any circuit remaining unaltered. From this and other circumstances we are led to infer that these circulations are not generalized velocity-components, but rather that the quantities $\kappa\rho$ are generalized momenta. Now the kinetic energy

* Read February 24, 1893.

of the system is naturally calculated as a function of the velocity-components of the solid and of these constant circulations (or the corresponding momenta); a form unsuited for obtaining the equations of motion. We ought either to have the kinetic energy expressed in terms of generalized velocity-components alone, or to know the "modified Lagrangian function" obtained by "ignoring" the velocity-components corresponding to the constant momenta or circulations. Either of these expressions involves constants which cannot be determined from the ordinary expression for the energy alone, and to determine them in the usual way it is necessary to resort to arguments based on a consideration of the "impulse" by which the motion might be set up from rest.

In the following investigation the equations of motion are deduced from purely hydrodynamical considerations, and from them the modified function is found. In §§ 12–16 the equations of motion are interpreted for the case in which the solid is a light rigid framework and the inertia is entirely due to the circulation of the liquid, and the results are applied to interpret the effective forces of the cyclic motion for a perforated solid in general.

General Hydrodynamical Equations.

2. Let a perforated solid bounded by the surface S be moving through an infinite mass of liquid (density ρ) with translational and rotational velocity-components u, v, w, p, q, r, referred to axes fixed in the solid, and let $\kappa_1, \kappa_2, \kappa_3 \ldots \kappa_m$ be the circulations in circuits drawn through the various apertures. Then we know that ϕ the velocity-potential of the fluid motion may be expressed as a linear function of the velocities and circulations in the form

$$\phi = u\phi_u + v\phi_v + w\phi_w + p\phi_p + q\phi_q + r\phi_r + \Sigma\kappa\phi_\kappa, \quad . \quad (1)$$

where evidently $\phi_u = \partial\phi/\partial u$ &c., and the coefficients $\phi_u \ldots$ depend only on the form of the solid.

If $d\nu$ denotes the element of the normal to S measured from the solid into the liquid, (l, m, n) its direction-cosines, then, in the usual way, we have

$$\frac{\partial\phi}{\partial\nu} = l(u - ry + qz) + m(v - pz + rx) + n(w - qx + py). \quad . \quad (2)$$

The six coefficients $\phi_u \ldots \phi_r$ are single-valued functions of the coordinates, while the coefficients ϕ_κ which determine the part of the velocity-potential due to the circulations are cyclic functions making $\partial\phi_\kappa/\partial\nu=0$ at the surface of the solid; these coefficients are supposed known for each form of solid, although their determination in any given case is generally beyond the range of mathematical analysis.

3. Let $\sigma_1, \sigma_2, \ldots \sigma_m$ be barriers drawn across the perforations; then, in the usual way, the kinetic energy of the liquid is found to be $\mathfrak{T}$, where

$$\mathfrak{T}=-\tfrac{1}{2}\rho\iint\phi\frac{\partial\phi}{\partial\nu}\,d\mathrm{S}+\tfrac{1}{2}\rho\Sigma\kappa\iint\frac{\partial\phi}{\partial\nu}\,d\sigma=\mathfrak{T}_1+\mathrm{K}. \quad . \quad (3)$$

Here $\mathfrak{T}_1$ is a quadratic function of the velocity-components of the solid, and is the kinetic energy when the motion is acyclic, and K is a quadratic function of the circulations.

If the axes were fixed in space, the pressure equation (supposing no forces to act on the liquid) would be

$$\frac{p_1}{\rho}+\frac{\partial\phi}{\partial t}+\tfrac{1}{2}q_1^2=\text{const.},$$

(where p_1 = pressure, q_1 = resultant velocity of liquid). Owing to the motion of the axes, however, $\partial\phi/\partial t$ must be replaced by the rate of change of ϕ at a fixed point, that is by

$$\frac{d\phi}{dt}-(u-yr+zq)\frac{\partial\phi}{\partial x}-(v-zp+xr)\frac{\partial\phi}{\partial y}-(w-xq+yp)\frac{\partial\phi}{\partial z},$$

whence the pressure equation becomes

$$\frac{p_1}{\rho}+\frac{d\phi}{dt}-(u-yr+zq)\frac{\partial\phi}{\partial x}-(v-zp+xr)\frac{\partial\phi}{\partial y}-(w-xq+yp)\frac{\partial\phi}{\partial z}$$
$$+\tfrac{1}{2}q_1^2=\text{const.} \quad . \quad . \quad . \quad . \quad (4)$$

The Mutual Reactions between the Solid and Liquid.

4. Let $\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1, \mathrm{L}_1, \mathrm{M}_1, \mathrm{N}_1$ be the component forces and couples which the solid exerts on the liquid; then we have evidently

$$\mathrm{X}_1=\iint l\,p_1\,d\mathrm{S}, \qquad \mathrm{L}_1=\iint(ny-mz)\,p_1\,d\mathrm{S}. \quad . \quad . \quad (5)$$

To reduce these expressions to the required form, we shall have to resort to repeated applications of Green's formula. Since the velocity-potential ϕ is a multiple-valued function,

it follows that in transforming volume integrals involving ϕ we shall obtain surface integrals over the barriers $\sigma_1, \sigma_2, \ldots \sigma_m$ as well as over S the surface of the solid. On the other hand, the pressure p_1 and the velocity-components $\partial\phi/\partial x$, $\partial\phi/\partial y$, $\partial\phi/\partial z$ are single-valued and do not contribute barrier terms to the surface integrals. Moreover, since the circulations κ are independent of the time,

$$\frac{\partial\phi}{\partial t} = \dot{u}\phi_u + \dot{v}\phi_v + \dot{w}\phi_w + \dot{p}\phi_p + \dot{q}\phi_q + \dot{r}\phi_r ;$$

and $\partial\phi/\partial t$ is therefore a single-valued function of the velocity-components of the solid satisfying Laplace's equation.

We also notice that

$$l = \frac{\partial\phi_u}{\partial\nu}, \quad ny - mz = \frac{\partial\phi_p}{\partial\nu}, \quad . \quad . \quad . \quad . \quad . \quad (6)$$

as may be at once seen by differentiating (2) with respect to u and p respectively.

5. Substituting for p_1 in (5) in terms of the velocities, we have

$$\begin{aligned}\frac{X}{\rho} = & -\iint l\frac{d\phi}{dt}\,dS \\ & + \iint\left\{(u - yr + zq)\frac{\partial\phi}{\partial x} + \text{(two similar)}\right\} l\,dS \\ & - \tfrac{1}{2}\iint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} l\,dS. \quad . \quad . \quad . \quad . \quad (7)\end{aligned}$$

The first line of this expression is, from (6), equal to

$$\begin{aligned} & -\iint\frac{d\phi}{dt}\frac{\partial\phi_u}{\partial\nu}\,dS \\ & = \iiint\left\{\frac{\partial\dot{\phi}}{\partial x}\frac{\partial\phi_u}{\partial x} + \frac{\partial\dot{\phi}}{\partial y}\frac{\partial\phi_u}{\partial y} + \frac{\partial\dot{\phi}}{\partial z}\frac{\partial\phi_u}{\partial z}\right\} dx\,dy\,dz\end{aligned}$$

by Green's transformation. Remembering that ϕ_u is independent of the time, this integral, taken throughout the liquid, becomes

$$\begin{aligned} & = \frac{d}{dt}\frac{\partial}{\partial u}\,\tfrac{1}{2}\iiint\left\{\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right\} dx\,dy\,dz \\ & = \frac{1}{\rho}\frac{d}{dt}\frac{\partial\mathfrak{T}}{\partial u} = \frac{1}{\rho}\frac{\partial}{\partial t}\frac{\partial\mathfrak{T}_1}{\partial u}.\end{aligned}$$

By Green's transformation the second line is equal to

$$-\iiint \frac{\partial}{\partial x}\left\{(u-yr+zq)\frac{\partial \phi}{\partial x} + \text{(two similar)}\right\} dx\,dy\,dz$$

$$= -\iiint\left(r\frac{\partial \phi}{\partial y} - q\frac{\partial \phi}{\partial z}\right) dx\,dy\,dz$$

$$-\iiint\left\{(u-yr+zq)\frac{\partial}{\partial x} + \text{(two similar)}\right\}\frac{\partial \phi}{\partial x} dx\,dy\,dz,$$

which by a second application of Green's transformation becomes

$$=\iint(mr-nq)\phi dS - \Sigma\kappa\iint(mr-nq)d\sigma$$

$$+\iint\{l(u-yr+zq) + m(v-zp+xr) + n(w-xq+yp)\}\frac{\partial \phi}{\partial x}dS$$

$$=\iint(mr-nq)\phi dS - \Sigma\kappa\iint(mr-nq)d\sigma + \iint\frac{\partial \phi}{\partial \nu}\frac{\partial \phi}{\partial x}dS$$

by (2).

Lastly, the third line of (7) is, by Green's transformation,

$$=\iiint\left\{\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial \phi}{\partial z}\frac{\partial^2 \phi}{\partial x \partial z}\right\} dx\,dy\,dz$$

$$=-\iint\frac{\partial \phi}{\partial \nu}\frac{\partial \phi}{\partial x}dS.$$

Hence, adding the several terms together, we have

$$X_1 = \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial u}$$

$$+ r\{\rho\iint m\phi dS - \Sigma\kappa\rho\iint m d\sigma\} - q\{\rho\iint n\phi dS - \Sigma\kappa\rho\iint n d\sigma\} \quad . \quad (8)$$

Now by (6),

$$\rho\iint m\phi dS = \rho\iint\frac{\partial \phi_v}{\partial \nu}\phi dS$$

$$=\Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \rho\iiint\left\{\frac{\partial \phi}{\partial x}\frac{\partial \phi_v}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \phi_v}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \phi_v}{\partial z}\right\} dx\,dy\,dz$$

$$=\Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \frac{\partial}{\partial v}\tfrac{1}{2}\rho\iiint\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\} dx\,dy\,dz$$

$$=\Sigma\kappa\rho\iint\frac{\partial \phi_v}{\partial \nu}d\sigma - \frac{\partial \mathfrak{T}}{dv}.$$

Therefore

$$\mathrm{X}_1 = \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial u} - r\left\{\frac{\partial \mathfrak{T}}{\partial v} + \Sigma\kappa\rho\iint\left(m - \frac{\partial \phi_v}{\partial \nu}\right)d\sigma\right\}$$

$$+ q\left\{\frac{\partial \mathfrak{T}}{\partial w} + \Sigma\kappa\rho\iint\left(n - \frac{\partial \phi_w}{\partial \nu}\right)d\sigma\right\}. \quad . \quad . \quad (9)$$

6. In like manner we have

$$\frac{\mathrm{L}_1}{\rho} = -\iint (ny - mz)\frac{d\phi}{dt}d\mathrm{S}$$

$$+\iint\left\{(u - yr + zq)\frac{\partial \phi}{\partial x} + (\text{two similar})\right\}(ny - mz)\,d\mathrm{S}$$

$$-\tfrac{1}{2}\iint\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right\}(ny - mz)\,d\mathrm{S}$$

$$= -\iint \frac{d\phi}{dt}\frac{\partial \phi_p}{\partial \nu}d\mathrm{S}$$

$$-\iiint\left\{y\left(q\frac{\partial \phi}{\partial x} - p\frac{\partial \phi}{\partial y}\right) - z\left(p\frac{\partial \phi}{\partial z} - r\frac{\partial \phi}{\partial x}\right)\right\}dx\,dy\,dz$$

$$-\iiint\left[y\left\{(u - yr + zq)\frac{\partial}{\partial x} + (\text{two similar})\right\}\frac{\partial \phi}{\partial z}\right.$$

$$\left. - z\left\{(u - yr + zq)\frac{\partial}{\partial x} + (\text{two similar})\right\}\frac{\partial \phi}{\partial y}\right]dx\,dy\,dz$$

$$+\iiint\left(\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial}{\partial z}\right)\left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y}\right)dx\,dy\,dz$$

$$= \frac{1}{\rho}\frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial p}$$

$$+\iint\left\{l(u - yr + zq) + (\text{two sim.}) - \frac{\partial \phi}{\partial \nu}\right\}\left(y\frac{\partial \phi}{\partial z} - z\frac{\partial \phi}{\partial y}\right)d\mathrm{S}$$

$$-\iiint\left[y\left(q\frac{\partial \phi}{\partial x} - p\frac{\partial \phi}{\partial y}\right) - z\left(p\frac{\partial \phi}{\partial z} - r\frac{\partial \phi}{\partial x}\right)\right.$$

$$\left. - (v - zp + xr)\frac{\partial \phi}{\partial z} + (w - xq + yp)\frac{\partial \phi}{\partial y}\right]dx\,dy\,dz.$$

Remembering that in this expression one factor of the surface integral is zero at every point of S, we have, by again applying Green's transformation to the volume integral,

$$
\begin{aligned}
L_1 = {} & \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial p} \\
& + w\{\rho\iint m\phi d\mathrm{S} - \Sigma\kappa\rho\iint m d\sigma\} - v\{\rho\iint n\phi d\mathrm{S} - \Sigma\kappa\rho\iint n d\sigma\} \\
& + r\{\rho\iint (lz - nx)\phi d\mathrm{S} - \Sigma\kappa\rho\iint (lz - nx) d\sigma\} \\
& - q\{\rho\iint (mx - ly)\phi d\mathrm{S} - \Sigma\kappa\rho\iint (mx - ly) d\sigma\} \quad . \quad . \quad (10)
\end{aligned}
$$

Now, just as before,

$$
\begin{aligned}
\rho\iint (lz - mx)\phi d\mathrm{S} & = \rho\iint \frac{\partial \phi_q}{\partial \nu}\phi d\mathrm{S} \\
& = \Sigma\kappa\rho\iint \frac{\partial \phi_q}{\partial \nu} d\sigma \\
& \qquad - \rho\iiint\left(\frac{\partial \phi}{\partial x}\frac{\partial \phi_q}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \phi_q}{\partial y} + \frac{\partial \phi}{\partial z}\frac{\partial \phi_q}{\partial z}\right) dx\, dy\, dz \\
& = \Sigma\kappa\rho\iint \frac{\partial \phi_q}{\partial \nu} d\sigma - \frac{\partial \mathfrak{T}}{\partial q}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
L_1 = {} & \frac{d}{dt}\frac{\partial \mathfrak{T}}{\partial p} \\
& - w\left\{\frac{\partial \mathfrak{T}}{\partial v} + \Sigma\kappa\rho\iint\left(m - \frac{\partial \phi_v}{\partial \nu}\right) d\sigma\right\} \\
& + v\left\{\frac{\partial \mathfrak{T}}{\partial w} + \Sigma\kappa\rho\iint\left(n - \frac{\partial \phi_w}{\partial \nu}\right) d\sigma\right\} \\
& - r\left\{\frac{\partial \mathfrak{T}}{\partial q} + \Sigma\kappa\rho\iint\left(lz - nx - \frac{\partial \phi_q}{\partial \nu}\right) d\sigma\right\} \\
& + q\left\{\frac{\partial \mathfrak{T}}{\partial r} + \Sigma\kappa\rho\iint\left(mx - ly - \frac{\partial \phi_r}{\partial \nu}\right) d\sigma\right\} \quad . \quad . \quad (11)
\end{aligned}
$$

Application to the Equations of Motion.

7. The equations of motion of the solid may now be written down at once. Let $\mathfrak{T}'$ be the kinetic energy of the solid, T the total kinetic energy $= \mathfrak{T} + \mathfrak{T}'$; and suppose that the motion takes place under the action of a system of external impressed forces and couples designated by X, Y, Z, L, M, N. Then the effective forces and couples to which the motion of the solid itself is due are $X - X_1 \ldots$, $L - L_1 \ldots$, respectively, and

the six equations of motion of the solid referred to the moving axes are of the form

$$\frac{d}{dt}\frac{\partial\mathfrak{T}'}{\partial u}-r\frac{\partial\mathfrak{T}'}{\partial v}+q\frac{\partial\mathfrak{T}'}{\partial w}=\mathrm{X}-\mathrm{X}_1, \quad . \quad . \quad . \quad (12)$$

$$\frac{d}{dt}\frac{\partial\mathfrak{T}'}{\partial p}-w\frac{\partial\mathfrak{T}'}{\partial v}+v\frac{\partial\mathfrak{T}'}{\partial w}-r\frac{\partial\mathfrak{T}'}{\partial q}+q\frac{\partial\mathfrak{T}'}{\partial r}=\mathrm{L}-\mathrm{L}_1. \quad . \quad (13)$$

Hence, on substitution, we see that the required equations of motion are found by writing T for $\mathfrak{T}$ and X, Y, Z, L, M, N for X_1, Y_1, Z_1, L_1, M_1, N_1 in equations (9) (11). The resulting equations may be written :

$$\mathrm{X}=\frac{d}{dt}\frac{\partial\mathrm{T}}{\partial u}-r\left(\frac{\partial\mathrm{T}}{\partial v}+\eta\right)+q\left(\frac{\partial\mathrm{T}}{\partial w}+\zeta\right), \quad . \quad . \quad . \quad . \quad . \quad (14)$$

$$\mathrm{L}=\frac{d}{dt}\frac{\partial\mathrm{T}}{\partial p}-w\left(\frac{\partial\mathrm{T}}{\partial v}+\eta\right)+v\left(\frac{\partial\mathrm{T}}{\partial w}+\zeta\right)$$

$$-r\left(\frac{\partial\mathrm{T}}{\partial q}+\mu\right)+q\left(\frac{\partial\mathrm{T}}{\partial r}+\nu\right), \quad (15)$$

where ξ, η, ζ, λ, μ, ν are defined by the equations

$$\xi=\Sigma\kappa\rho\iint\left(l-\frac{\partial\phi_u}{\partial\nu}\right)d\sigma, \text{ \&c.} \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$\lambda=\Sigma\kappa\rho\iint\left(ny-mz-\frac{\partial\phi_p}{\partial\nu}\right)d\sigma, \text{ \&c.} \quad . \quad . \quad . \quad (17)$$

As Lamb has pointed out ('Motion of Fluids,' p. 140), the ix quantities (ξ, η, ζ, λ, μ, ν) are "the components of the impulse of the cyclic fluid motion which remains when the solid is (by forces applied to it alone) brought to rest"*. They are linear functions of the circulations and their form depends on the form of the solid. If there is only one aperture they are all proportional to the circulation κ.

The Modified Lagrangian Function.

8. We shall now show that the motion of the solid can be determined in terms of Routh's modified Lagrangian function,

* Our ξ, η, ζ, λ, μ, ν are the same as the ξ_0, η_0, ζ_0, λ_0, μ_0, ν_0 of Lamb, or the $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, $\mathfrak{L}$, $\mathfrak{M}$, $\mathfrak{N}$ of Basset's 'Hydrodynamics.'

and shall find the form of this function for the system. Putting

$$H = T + \xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r + F(\kappa\rho), \quad . \quad . \quad (18)$$

where $F(\kappa\rho)$ denotes any function whatever of the quantities $\kappa\rho$, we see that the equations of motion reduce to the standard form

$$X = \frac{d}{dt}\frac{\partial H}{\partial u} - r\frac{\partial H}{\partial v} + q\frac{\partial H}{\partial w}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

$$L = \frac{d}{dt}\frac{\partial H}{\partial p} - w\frac{\partial H}{\partial v} + v\frac{\partial H}{\partial w} - r\frac{\partial H}{\partial q} + q\frac{\partial H}{\partial r}. \quad . \quad (20)$$

The function H, therefore, plays the same part in determining the equations of motion of the solid as the kinetic energy T in the case of an imperforated solid (or any solid when the motion of the liquid is acyclic). It remains (i.) to determine what quantities are to be regarded as the generalized velocities if the quantities $\kappa\rho$ are regarded as generalized momenta; (ii.) to find the form of the function $F(\kappa\rho)$ in order that H may represent the modified Lagrangian function.

9. Let $\dot{\chi}_m$ be the generalized velocity-component corresponding to the ignored momentum $\kappa_m\rho$. Then, as Routh has shown ('Rigid Dynamics,' vol. i. § 420), the modified Lagrangian function H is of the form

$$H = T - \Sigma\kappa_m\rho\dot{\chi}_m, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21)$$

and therefore by equating the two expressions for H we must have

$$\xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r + F(\kappa) = -\Sigma\kappa\rho\dot{\chi}. \quad . \quad . \quad (22)$$

Since $\dot{\chi}_m$ is the generalized velocity-component corresponding to the momentum $\kappa_m\rho$, therefore

$$\frac{\partial H}{\partial\,.\,\kappa_m\rho} = -\dot{\chi}_m. \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Now H is a homogeneous quadratic function of the six velocities $(u\ldots, p\ldots)$ and the momenta $\kappa\rho$; therefore

$$2H = \Sigma u\frac{\partial H}{\partial u} + \Sigma\kappa\rho\frac{\partial H}{\partial\,.\,\kappa\rho} = \Sigma u\frac{\partial H}{\partial u} - \Sigma\kappa\rho\dot{\chi}. \quad . \quad (24)$$

Hence, from (21),

$$2T = \Sigma u \frac{\partial H}{\partial u} + \Sigma \kappa\rho\dot{\chi} = \Sigma u \frac{\partial H}{\partial u} - \Sigma \kappa\rho \frac{\partial H}{\partial \cdot \kappa\rho}. \quad . \quad . \quad (25)$$

The portions of T and H which involve only the momenta $\kappa\rho$, and are independent of the six velocities $(u \ldots, p \ldots)$, must arise from the terms $\Sigma\kappa\rho\dot{\chi}$ in the above expressions (24) (25), and must therefore be equal and of opposite sign in the expressions T and H respectively. Hence, since from (3)

$$T = \mathfrak{T}' + \mathfrak{T}_1 + K,$$

the portion of H which is independent of the six velocities $(u \ldots, p \ldots)$ must be $-K$, so that

$$\begin{aligned} H &= \mathfrak{T}' + \mathfrak{T}_1 + (\xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r) - K, \\ &= T + (\xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r) - 2K, \quad . \quad . \quad (26) \end{aligned}$$

and therefore $$F(\kappa\rho) = -2K. \quad . \quad . \quad . \quad . \quad . \quad . \quad (27)$$

The function $F(\kappa\rho)$ does not enter into the six equations of motion of the solid, but its form requires to be determined if we wish to reduce the equations of motion of the whole system to the canonical or Hamiltonian form.

The Generalized Velocities and Momenta.

10. Comparing (21) with (27), we see that

$$\Sigma\kappa\rho\dot{\chi} = 2K - (\xi u + \eta v + \zeta w + \lambda p + \mu q + \nu r). \quad . \quad (28)$$

Now equation (3) may be written in the form

$$2(\mathfrak{T}_1 + K) = -\rho \iint \phi \frac{\partial}{\partial \nu}(u\phi_u + \ldots + p\phi_p + \ldots + \Sigma\kappa\phi_\kappa) dS$$
$$+ \Sigma\kappa\rho \iint \frac{\partial}{\partial \nu}(u\phi_u + \ldots + p\phi_p + \ldots + \Sigma\kappa\phi_\kappa) d\sigma. \quad (29)$$

But by § 2, $\partial\phi_\kappa/\partial\nu = 0$ all over the surface S of the solid. Hence, equating the terms independent of the six velocities $(u \ldots, p \ldots)$ on the two sides of (29), we have

$$2K = \Sigma\kappa\rho \iint \frac{\partial}{\partial \nu}(\Sigma\kappa\phi_\kappa) d\sigma. \quad . \quad . \quad . \quad . \quad (30)$$

But by (16) (17),

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r$$

$$=-\Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(u\phi_u+\ldots+p\phi_p+\ldots)\,d\sigma$$

$$+\Sigma\kappa\rho\iint\{lu+mv+nw+(ny-mz)p+\ldots\}\,d\sigma.$$

Hence, by (28) and (30),

$$\Sigma\kappa\rho\dot{\chi}=\Sigma\kappa\rho\iint\frac{\partial}{\partial\nu}(u\phi_u+\ldots+p\phi_p+\ldots+\Sigma\kappa\phi_\kappa)\,d\sigma$$

$$-\Sigma\kappa\rho\iint\{l(u-yr+zq)+m(v-zp+xr)+n(w-xq+yp)\}\,d\sigma;$$

and therefore

$$\dot{\chi}_m=\iint\left(\frac{\partial\phi}{\partial\nu}-l(u-yr+zq)-(\text{two similar})\right)d\sigma_m. \quad . \quad (31)$$

Now $\partial\phi/\partial\nu$ is the velocity of the liquid resolved along the normal to the barrier σ_m, and

$$l(u-yr+zq)+m(v-zp+xr)+n(w-xq+yp)$$

is the velocity of the barrier σ_m resolved normally to itself at the point x, y, z, supposing the barrier to be fixed relatively to the solid. Their difference, therefore, represents the normal relative velocity of the liquid with respect to the barrier.

Hence $\dot{\chi}_m$ represents the total rate of flow of the liquid across the barrier σ_m relative to the solid; in other words, the generalized velocity corresponding to the ignored momentum $\rho\kappa_m$ is the volume of liquid per unit time flowing through the aperture relatively to the solid.

This property is proved in a different way by Basset in his 'Hydrodynamics,' vol. i. page 176.

The Form of the Modified Function.

11. It may be interesting to examine a little more closely the effect of the circulations on the motion of a solid.

When the motion of the liquid is acyclic, the kinetic energy is a homogeneous quadratic function of (u, v, w, p, q, r). In general it therefore involves 21 constants, but by a suitable choice of axes it is *always* possible to reduce this number by

six, and a further reduction may be effected when the body is symmetrical. When the motion is cyclic the kinetic energy must be replaced by the modified function which in addition contains the seven terms

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r-\mathrm{K},$$

of which the last term does not enter into the equations of motion of the solid. The six coefficients ($\xi, \ldots \lambda, \ldots$) are linear functions of the circulations, and they remain constant so long as only conservative forces act on the liquid, for the circulations themselves then remain constant. Hence the modified function H may be regarded as a non-homogeneous quadratic function of the six velocities involving 28 constants, of which 27 enter into the six equations of motion of the solid.

Equations of Motion of a light thin framework of rigid wires.

12. To take the simplest possible case, let us suppose the solid to consist of a network of infinitely thin rigid massless wires through the meshes of which the liquid is circulating. If the motion of the liquid were acyclic, the wires would simply cut through the liquid without setting it in motion: hence the kinetic energy $\mathfrak{T}'+\mathfrak{T}_1$ of the acyclic motion vanishes, and the modified function becomes

$$\mathrm{H}=\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r-\mathrm{K}, \quad . \quad . \quad . \quad (32)$$

a result otherwise evident from the fact that $\mathfrak{T}_1$ only involves surface integrals taken over the infinitely small surface of the solid, while $\xi, \eta, \zeta, \lambda, \mu, \nu$ being integrals taken over the finite surfaces of barriers are in general finite.

If we choose as our axis of x the Poinsot's central axis of the impulse whose six components are $\xi, \eta, \zeta, \lambda, \mu, \nu$, the modified function will reduce to the form

$$\mathrm{H}=\Xi u+\Lambda p-\mathrm{K}. \quad . \quad . \quad . \quad . \quad . \quad (33)$$

If there is only one aperture, $\xi, \eta, \zeta, \lambda, \mu, \nu$ are all proportional to the circulation κ and the central axis of the impulse is fixed in position relative to the solid: if there are several apertures the position of the axis depends on the ratios of the circulations through the various apertures, but

throughout the motion it in every case remains fixed relatively to the solid.

The six equations of motion (19) (20) now reduce to

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi+r\Lambda, \\ Z=-q\Xi, & N=-v\Xi-q\Lambda. \end{array}\right\} \quad . \quad . \quad . \quad (34)$$

Since these equations do not involve u or p, we see that *no forces will have to act on the solid in order to maintain a screw motion whose axis coincides with the central axis of the impulse.*

13. To interpret the equations still further, let us suppose that u and p are both zero, since they do not enter into the equations of motion. Then the motion whose components are $(0, v, w, 0, q, r)$ consists of two screws whose axes are the axes of y and z respectively, and, by the theory of screws, these are equivalent to a single screw whose axis is a certain straight line intersecting the axis of x and perpendicular to it. We may take this straight line as our axis of z, for hitherto we have only fixed the position of the axis of x. We have then

$$v=0, \quad q=0.$$

The equations (34) therefore reduce to

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi+r\Lambda, \\ Z=0, & N=0. \end{array}\right\} \quad . \quad . \quad . \quad (35)$$

Hence the solid is acted on by a wrench (Y, M) whose axis is the axis of y. Thus the axis of the impressed wrench is perpendicular to the central axis of the impulse of the fluid motion, and to the axis of the screw motion of the body.

Let Π be the pitch of the impulse, ϖ the pitch of the screw motion of the solid, P the pitch of the impressed wrench, then

$$\Pi=\frac{\Lambda}{\Xi}, \quad \varpi=\frac{w}{r}, \quad P=\frac{M}{Y},$$

and therefore by (35),

$$P=\varpi+\Pi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36)$$

is the relation connecting the three pitches.

In particular, if $r=0$ the equations of motion give

$$Z=0, \quad M=w\Xi,$$

showing that a couple M about the axis of y will produce translational motion with velocity M/Ξ along the axis of z.

14. More generally, let the motion be a screw motion about an axis whose inclination to the axis of x is θ and whose shortest distance from that axis is a. Take this shortest distance as the axis of y, and let the screw motion consist of a linear velocity V combined with an angular velocity Ω, the pitch V/Ω being denoted, as before, by ϖ.

It will be readily found that the six components of the screw motion are

$$\left.\begin{array}{ll} u=V\cos\theta+\Omega a\sin\theta, & p=\Omega\cos\theta, \\ v=0, & q=0, \\ w=V\sin\theta-\Omega a\cos\theta, & r=\Omega\sin\theta, \end{array}\right\} ; \quad . \quad . \quad (37)$$

so that the equations (34) now give

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=\Omega\Xi\sin\theta, & M=V\Xi\sin\theta-\Omega a\Xi\cos\theta+\Omega\Lambda\sin\theta, \\ Z=0, & N=0, \end{array}\right\} . (38)$$

The impressed wrench therefore has for its axis the shortest distance between the axis of the screw motion of the solid and the axis of the impulse of the cyclic fluid motion. To find the pitch of the wrench, we have, by division,

$$\frac{M}{Y}=\frac{V}{\Omega}-a\cot\theta+\frac{\Lambda}{\Xi},$$

that is,

$$P=\varpi-a\cot\theta+\Pi. \quad . \quad . \quad . \quad . \quad . \quad (39)$$

15. In the case of a fine massless circular ring Λ vanishes, or the impulse of the cyclic motion is purely translational. For it is clear that the axis of the ring is the axis of this impulse (the above axis of x), also the fluid motion will evidently be unaffected by rotating the ring about its axis; and therefore the modified function is independent of the angular velocity p.

The equations (34) now become

$$\left.\begin{array}{ll} X=0, & L=0, \\ Y=r\Xi, & M=w\Xi, \\ Z=-q\Xi, & N=-v\Xi. \end{array}\right\} \quad . \quad . \quad . \quad (40)$$

Hence a constant force Y along the axis of y causes uniform rotation with angular velocity Y/Ξ about the axis of z, and a constant couple M about the axis of y causes uniform translational velocity M/Ξ along the axis of z.

It is to be noticed that the impressed wrench never does work in the resulting screw motion, in accordance with the principle of Conservation of Energy.

16. The above results show the effective forces produced by circulation of the fluid on any perforated solid whatever. In the general case the modified function contains the quadratic terms $\mathfrak{T}'+\mathfrak{T}_1$ in addition to the terms of the first degree considered in the above investigation. If we suppose that the solid is moving in any given manner, the six equations of motion (19, 20) determine the components of the impressed wrench (X, Y, Z, L, M, N) necessary to maintain the given motion. This impressed wrench may be divided into two parts, one being due to the terms $\mathfrak{T}'+\mathfrak{T}_1$ in the modified function, the other being due to the terms

$$\xi u+\eta v+\zeta w+\lambda p+\mu q+\nu r.$$

The first portion is the same as if the motion were acyclic, and represents, therefore, the wrench which would have to be impressed on the solid in order to maintain the given motion if there were no circulation. The second part represents the additional wrench which must be applied on account of the circulations, and the equations to determine it are of the forms found above.

We notice, in particular, that if the solid has any screw motion whose axis coincides with the axis of the impulse of the cyclic fluid motion, the latter wrench vanishes; so that the forces required to maintain the motion are unaffected by the circulations. In other cases the additional wrench is about an axis perpendicular to the axis of the impulse. This is true whatever be the form of the solid and the number of the circulations; but, as has already been pointed out, the position of the axis of the impulse relative to the solid is not in general independent of the circulations unless the solid has but a single aperture.

It is probable that these results might be made to furnish mechanical illustrations of certain physical phenomena; but with these we are not concerned in the present paper.

Note on the foregoing Paper.

Concerning the proper measurement of the impulse of the cyclic motion, a difficulty arises; for, as Mr. Bryan remarks, this motion cannot be set up from rest by impulses applied to the solid alone. Suppose, however, that we close each perforation by a barrier in the usual way, and let the barriers be acted on by the impulsive pressures $\kappa_1\rho$, $\kappa_2\rho$, ... respectively. And instead of these impulsive pressures being due to external forces, *suppose that they are due to some immaterial mechanism attached to the solid.* In general, an impulsive wrench must act on the solid to keep it at rest, and *this wrench is the required impulse of the cyclic motion*; for the only other impulses acting on the system are due to the mutual reactions of the solid and fluid, exerted partly over the surface of the solid and partly through the barriers and attached mechanism, and such mutual reactions cannot affect the impulse. The wrench thus found is of course the same as would be obtained by supposing the impulses on the barriers to be due to *external* impulsive forces, and compounding with these the impulse *then* necessary to hold the solid at rest. This is in agreement with Prof. Lamb's investigation, which Mr. Bryan has quoted.

More generally, if the solid is in motion, and the liquid is also circulating, we may suppose the instantaneous motion to have been set up from rest by an immaterial mechanism connecting the barriers with the solid at the same time that the requisite external impulses act on the solid. The resultant of these last is, as before, the impulse of the whole motion, and is identical with that found by supposing the barriers actuated by impulses *from without*, and compounding with these the impulse *then* necessary to give to the solid its instantaneous motion.

The same point may be further illustrated by supposing the circulations κ to vary continuously during the motion. To effect this variation we may suppose finite uniform pressures, $P_1 \dots P_m$, to be exerted over certain ideal surfaces which occupy the positions of barriers. The rate of variation $\dot{\kappa}$ of any circulation is given by $P = \dot{\kappa}\rho$, and in order that it may

take place without the direct operation of external forces and couples we may conceive the pressure P to be due, as before, to some highly idealized mechanism attached to the solid. As before, the only forces capable of modifying the impulse are the external forces acting on the solid ; and the equations of motion are therefore still to be found by equating the impressed force- and couple-components to the corresponding variations of the "impulse." Since we know the expressions for the impulse-components corresponding to a given instantaneous motion of the solid and given circulations, we have only to remember that in these expressions the κ's are functions of the time, and, just as before, the equations of motion are directly deducible from Hayward's formulæ. Equations (19) (20) of Mr. Bryan's paper will thus be applicable to the present case, provided that in the value of H given by (27) the κ's are allowed to vary.

An investigation proceeding from a consideration of the impulse of the whole motion is not so entirely satisfactory, I think, as the direct method given by Mr. Bryan ; but, at the same time, this brief attempt to interpret the impulse of the cyclic motion may not be without interest.—C. V. BURTON.

Note added by the Author.

Dr. Burton's note is of much value as showing more exactly what is meant by the "impulse" of the motion in the ordinary investigations given by Prof. Lamb, and, in a less intelligible form, by Basset.

The equations of motion under finite forces may be deduced by equating the change of momentum in a small time-interval δt to the impulse of the impressed forces, taking into account the fact that in the interval δt the origin has a displacement of translation $(u\delta t, v\delta t, w\delta t)$ and the axes have rotational displacements $(p\delta t, q\delta t, r\delta t)$, so that the final momenta are referred to a different set of axes to the original momenta.

The mode of forming the equations of motion is given by Prof. Greenhill (*Encyclopædia Britannica*, art. "Hydromechanics") for the case of acyclic motion, but it is hardly so obvious *why* in thus forming the equations of motion of a

perforated solid, it is necessary to include in the "impulse" terms representing the components of the wrench applied to the *barriers* as well as to the solid. We may, however, suppose the changes which actually occur in the time δt to have been produced as follows :—

1st. Let the solid and fluid be reduced to rest by an impulsive wrench applied to the solid, and transmitted to a series of barriers crossing the perforations. The components of this wrench will be found to be

$$\frac{\partial T}{\partial u} + \xi, \text{ \&c.}, \ldots \frac{\partial T}{\partial p} + \lambda, \text{ \&c.} \ldots$$

2nd. The barriers being rigidly connected with the solid, let the latter receive small displacements whose translational and rotational components are ($u\delta t$, $v\delta t$, $w\delta t$, $p\delta t$, $q\delta t$, $r\delta t$) and let the solid come to rest in its new position. The fluid will evidently also come to rest, and therefore no impulse will be impressed on the system by this change (as may be otherwise seen by supposing the change to take place very slowly).

3rd. Let the solid be set in motion with velocity-components ($u + \partial u/\partial t \,.\, \delta t, \ldots \quad p + \partial p/\partial t \,.\, \delta t \ldots$) referred to the new positions of the axes, and let the circulations κ be started in the new position of the solid by a suitable impulsive wrench applied to the solid and transmitted from it to the barriers.

Then the impulse of the impressed forces (components $X\delta t \ldots$, $L\delta t \ldots$) is the resultant of the wrenches required to stop the whole system in the first process and to start it again in the third.

It is, therefore, that impulse which must be compounded with the total impulse in the initial position in order to obtain the total impulse in the final position.

Whence Hayward's equations of motion follow at once (as shown in Greenhill's article above referred to), and they take the form of the above equations (14), (15).

If we were merely to stop the solid in the first process without stopping the liquid, the cyclic motion would cause the liquid to exert a pressure on the solid in the second process, and the impulse of this pressure would not be zero, but would have to be taken into account in forming the equations

of motion. It would be wrong, therefore, to deduce the equations of motion from the impulse applied to the solid alone, as is evident in the analogous case of a solid containing one or more gyrostats.

DISCUSSION.

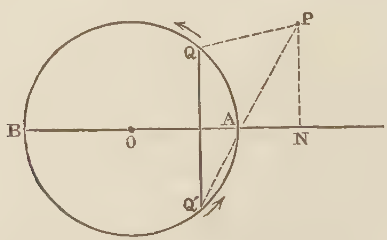
The President said the author had done good service by attacking the difficult problem by elementary methods. He had also arrived at some very interesting conclusions, particularly the one showing that a perforated body moving through a liquid required no force to keep up the motion.

XIII. *The Magnetic Field of a Circular Current.* *By* Professor G. M. MINCHIN, *M.A.**

CLERK MAXWELL gives a method of drawing the lines of magnetic force due to a circular current ('Electricity and Magnetism,' Art. 702) by means of a series of circles and a series of parallel lines. The object of the following paper is to show how these curves can be described by a slightly different method, and to exhibit the geometrical connexion of the series of circles.

Let AQBQ′ be the circular current whose sense is indicated by the arrows, the plane of the circle being that of the paper;

Fig. 1.



let P be any point in space, PN the perpendicular from P on the plane of the circle, and NAOB the diameter of the circle drawn through N. We shall calculate the *vector potential* of the current at P.

* Read March 10, 1893.

Draw any ordinate, QQ′, of the circle perpendicular to BA; and consider two equal elements of length of the circle, each equal to ds, at Q and Q′. Resolving each of these along and perpendicular to QQ′, we see that the latter components are in opposite senses, and hence their vector potentials at P cancel each other, since PQ = PQ′. If ψ is the angle QOA, a = radius of circle, i = current strength, the components of ids along Q′Q being equal and in the same sense, the two elements of current at Q′ and Q conspire in giving a vector potential $\frac{2i\cos\psi}{\text{PQ}}.ds$ perpendicular to the plane PON. Hence the total vector potential at P is perpendicular to the plane PON. If, therefore, OA is the axis of x, the perpendicular at O to the plane of the circle the axis of z, and the diameter at O perpendicular to AB the axis of y, the components of the vector potential being, as usual, denoted by F, G, H, the only component existing is G; but, by taking the components of the vector potential at a point indefinitely close to P in the direction of the axis of y, we easily find that

$$\frac{d\text{F}}{dy} = -\frac{\text{G}}{\alpha}.$$

Hence if X, Y, Z are the components of the force of the current per unit magnetic pole at P, since this force is the curl of the vector potential, we have

$$\text{X} = -\frac{d\text{G}}{d\gamma}, \quad \text{Y} = 0, \quad \text{Z} = \frac{d\text{G}}{d\alpha} + \frac{\text{G}}{\alpha},$$

where α (=ON) and γ (=NP) are the coordinates of P. If along the line of force at P the increments of the co-ordinates are $\Delta\alpha$, $\Delta\gamma$, we have

$$\frac{\Delta\alpha}{\Delta\gamma} = \frac{\text{X}}{\text{Z}}.$$

Hence along this line we have

$$\frac{d\text{G}}{d\alpha}\Delta\alpha + \frac{d\text{G}}{d\gamma}\Delta\gamma + \frac{\text{G}}{\alpha}\Delta\alpha = 0,$$

i.e., $\text{G}.\alpha =$ constant along the line of force.

We shall therefore calculate the vector potential, G, at P.

Evidently

$$G = 2i\int_0^\pi \frac{\cos\psi}{PQ}\,.\,ds$$

$$= 2ia\int_0^\pi \frac{\cos\psi\,.\,d\psi}{\sqrt{\alpha^2+a^2+\gamma^2-2a\alpha\cos\psi}}$$

$$= \frac{i}{\alpha}\int_0^\pi \left(\frac{\alpha^2+a^2+\gamma^2}{D}-D\right)d\psi,$$

denoting the denominator by D. Now let $\psi=\pi-\omega$, and let $\rho^2=(\alpha+a)^2+\gamma^2$, $\rho'^2=(\alpha-a)^2+\gamma^2$, so that $\rho=PB$, $\rho'=PA$. Then

$$G = \frac{i}{2\alpha}\int_0^\pi \left\{\frac{\rho^2+\rho'^2}{\rho}\,.\,\frac{1}{\Delta}-2\rho\Delta\right\}d\omega,$$

where

$$\Delta \equiv \sqrt{1-\left(1-\frac{\rho'^2}{\rho^2}\right)\sin^2\frac{\omega}{2}}.$$

Let $\omega=2\phi$, and $k^2=1-\frac{\rho'^2}{\rho^2}$; then, finally,

$$G = \frac{4ia}{\rho}\left\{\frac{2}{k^2}(K-E)-K\right\},$$

where K and E are the complete elliptic integrals of the first and second kinds with modulus k; so that the quantity in brackets is a function of the ratio $\frac{PA}{PB}$ simply.

Also, since $\rho^2-\rho'^2=4a\alpha$, we have $\alpha=\frac{\rho^2k^2}{4a}$, and the quantity $G\,.\,\alpha$ which is constant along the line of force is given by the equation

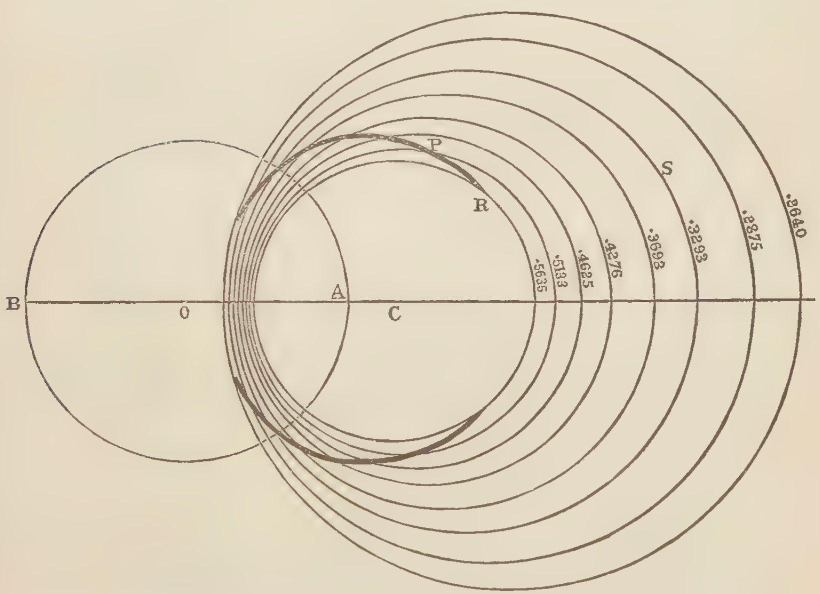
$$G\,.\,\alpha=i\rho\{2(K-E)-k^2K\}.$$

It is thus seen that at every point in space G is of the form $\frac{1}{\rho}f\left(\frac{\rho'}{\rho}\right)$; so that at all points on the surface for which $\frac{\rho'}{\rho}$ is a constant, the value of G will vary inversely as ρ. The surface for which $\frac{\rho'}{\rho}$ is constant is a sphere having its centre on the line BA produced and cutting the sphere having BA

for diameter orthogonally. If we assign a series of values to the ratio $\frac{\rho'}{\rho}$, we obtain a series of spheres having their centres on BA and cutting the given sphere orthogonally, the radius of each sphere of the series being, therefore, the length of a tangent from its centre to the sphere described on BA; for, given the base, BA, of a triangle, and the ratio of the sides, the locus of the vertex is a circle whose diameter is the join of the points which divide BA internally and externally in the given ratio. The surface locus of the vertex is the sphere generated by the revolution of this circle.

On account of the symmetry of the current round its axis through O, the lines of force and those of constant vector potential are the same in all planes through the axis.

Fig. 2.



We may, then, confine our attention to the plane PON, and suppose fig. 2 to be in this plane, the current being in this figure represented in projection by the line BA. Describe a series of circles having their centres on BA pro-

duced and cutting the circle described on BA as diameter orthogonally. Along each of these circles, then, the ratio $\frac{\text{PA}}{\text{PB}}$ is constant, P being any point on the circle.

Consider first the lines of constant vector potential. For each of the circles let the value of the quantity $\frac{2}{k^2}(\text{K}-\text{E})-\text{K}$ be calculated. Denote this quantity by Q for any one circle; then

$$\text{G} = \frac{4\text{Q}ia}{\rho};$$

so that if we wish to trace out the line of constant vector potential for which G has any given value, we can find the point, P, in which it cuts any circle of the series by measuring the length PB such that

$$\text{PB} = \frac{4\text{Q}i}{\text{G}} \cdot a.$$

Let PT be any circle of the co-orthogonal series cutting BA at n and m. Then for this circle

$$\frac{\rho'}{\rho} = \frac{\text{A}n}{\text{B}n} = \frac{m\text{A}}{m\text{B}};$$

and if this ratio is denoted by s, it is well known that

$$\frac{\text{CA}}{\text{CB}} = s^2,$$

where C is the centre of the circle. Now the modulus, k, of the elliptic integrals which belongs to the circle $m\text{P}n$ is $\left(1-\frac{\text{PA}^2}{\text{PB}^2}\right)^{\frac{1}{2}}$, *i.e.* $k^2=1-s^2$; hence

$$k^2 = \frac{\text{AB}}{\text{BC}},$$

or the square of the modulus is inversely proportional to the distance, BC, of the centre of the circle from B.

The circles employed by Clerk Maxwell in drawing the lines of force can be easily shown to be this co-orthogonal system whose centres are ranged along BA produced. For, his rule is to assign a series of values to θ, and construct

a series of circles whose centres lie on BA, the radius of each being $\frac{a}{2}(\operatorname{cosec}\theta - \sin\theta)$, while the distance of its centre from O is $\frac{a}{2}(\operatorname{cosec}\theta + \sin\theta)$; the modulus belonging to this circle is $\sin\theta$. For the series of circles he then calculates the values of the expression (constant for each circle) $\frac{\sin\theta}{(K-E)^2}$, and the point on each circle which lies on any assigned line of force is found by drawing a certain right line perpendicular to BA. It is at once found that this series of circles is precisely the co-orthogonal system above described; but Clerk Maxwell's modulus is not the same function of the ratio $\frac{\rho'}{\rho}$, or of the radius of the circle selected, as that adopted above; for, with Clerk Maxwell, if r is the radius of any circle of the series and k the corresponding modulus,

$$r = \frac{a}{2}\left(\frac{1}{k} - k\right),$$

whereas above we have

$$r = \frac{2\sqrt{1-k^2}}{k^2} \cdot a.$$

Of course (as stated in a note by Clerk Maxwell) the elliptic integrals depending on the one modulus can be transformed into elliptic integrals depending on the other; and in this case the transformation is the well-known one of Lagrange. But the constructions for the points in which any given line of force cuts the series of circles will not be the same in both cases—those of Clerk Maxwell depending on a series of right lines perpendicular to BA, and those above indicated depending on a series of radial distances from B.

When we propose to draw the line of constant vector potential through any point, P, which lies on a circle whose constant is Q_0, let PB be ρ_0; then the point, R, in which this line meets any other circle, whose constant is Q, is found from the relation

$$\rho = \rho_0 \frac{Q}{Q_0},$$

where $\rho = \mathrm{BR}$.

This latter method has a certain advantage for the eye, inasmuch as it enables us to see readily those circles of the series outside which the line of constant vector potential through any proposed point lies.

Consider now the lines of force. With the above value of Q, the quantity which is constant along a line of force is $\rho . k^2 Q$, so that on each of the above circles in fig. 2 we must now mark the number $k^2 Q$. Denote this by Q'. Then the above relation for points on the same line of constant vector potential becomes for the lines of force

$$\rho = \rho_0 \frac{Q_0'}{Q'},$$

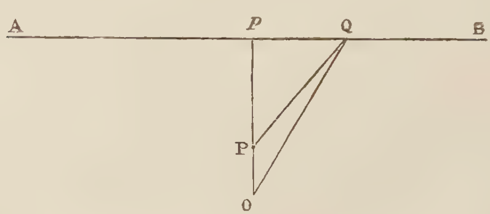
and the construction proceeds in the same way. The constants, Q', for the above series of circles, beginning at the innermost, are :—

·4841 ; ·4301 ; ·3775 ; ·3396 ; ·2782 ; ·2376 ; ·1954 ; ·1727.

The values of the Q's diminish outwards for the circles ; so that if we consider the line of vector potential at any point, S, suppose, which is such that SB is greater than the distance from B of the point along AC in which any circle interior to that passing through S cuts the line BAC, it is at once obvious that the line of vector potential which belongs to S is wholly outside all such circles. The numerical values of Q for the circles in fig. 2 are marked at the circumferences, and as much of the line of potential belonging to P is drawn as is justified by the number of circles represented in the figure.

The fundamental proposition of electromagnetism is that

Fig. 3.



the intensity of magnetic force produced at any point in presence of electric currents is the curl of the vector potential at the point. But if in the field there is a current in an

infinitely long straight wire, AB, we find that at every point in the field the vector potential due to this current is infinite. Hence it seems impossible to deduce the magnetic force, and the lines of magnetic force, from the above fundamental proposition. This result is unsatisfactory, and it manifestly points to some defect in our definition of the vector potential.

We are presented with a similar unsatisfactory result in the general theory of gravitation potential. Thus, taking the common definition of gravitation potential, if AB is a limited uniform bar attracting according to the law of inverse square, we know that the potential which it produces at any point, P, is proportional to $\log\left(\cot\frac{A}{2}\cot\frac{B}{2}\right)$, where $A = \angle PAB$, $B = \angle PBA$. Now, if the rod extends to infinity, this expression becomes infinite. I have shown ('Statics,' vol. ii. Art. 332) how this difficulty arises, and how it is to be remedied by mending the definition of potential. The difficulty is avoided in a similar manner with regard to the vector potential.

Thus, since we are concerned only with differential coefficients of the vector potential, the ordinary components, F, G, H, of this vector may have added to them any *constant* quantities whatever. This amounts to saying that the vector potential at any variable point, P, in the field is the vector potential at any *fixed* point, O, plus the vector difference between P and O. It does not matter whether the vector at O is infinite or not: it is a constant in the field. As in the general gravitation field we are concerned with *differences* of potential only, so in the electromagnetic field we are concerned with vector differences only.

Let us, then, calculate for the infinite straight current AB the vector difference between P and a point O on the perpendicular, Pp, at a constant distance O$p = a$ from the line.

Let P$p = r$, and let an element, ds, of the line AB be taken at any point, Q; let $\angle p\mathrm{PQ} = \theta$. Then the vector difference, due to this element, at P is $\frac{ds}{\mathrm{QP}} - \frac{ds}{\mathrm{QO}}$, or

$$\left\{1 - \frac{r}{\sqrt{r^2 + (a^2 - r^2)\cos^2\theta}}\right\} \cdot \frac{d\theta}{\cos\theta}.$$

Double the integral of this from $\theta=0$ to $\theta=\frac{\pi}{2}$ is the vector difference at P due to unit current in AB. Expanding the radical in ascending powers of $\lambda\left(\equiv\frac{a^2-r^2}{r^2}\right)$, we have the vector equal to

$$2\int_0^{\frac{\pi}{2}}\left\{\tfrac{1}{2}\lambda-\frac{1\,.\,3}{2\,.\,4}\lambda^2\cos^2\theta+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\lambda^3\cos^4\theta-\ \ldots\right\}d(\sin\theta)$$

$$=\lambda-\tfrac{1}{2}\lambda^2+\tfrac{1}{3}\lambda^3-\tfrac{1}{4}\lambda^4+\ \ldots$$

and this $=\log_e(1+\lambda)=2\log\frac{a}{r}$. Thus, then, the vector difference at any point, P, is measured by

$$C-2\log r,$$

where C is a constant; and this gives the known value of the magnetic force at P, viz., $-\frac{dG}{dr}$ (where G is the vector potential), perpendicular to the plane PAB, *i.e.* $\frac{k}{r}$, where k is a constant. In this way, then, the inconvenience of dealing with an infinite vector potential in presence of an infinitely long (or very long) straight current is avoided.

The lines of constant magnetic potential, or the loci of points, P, at which the given circular current subtends a constant conical ("solid") angle, are the orthogonal trajectories of the lines of force, and can be drawn when these lines are drawn.

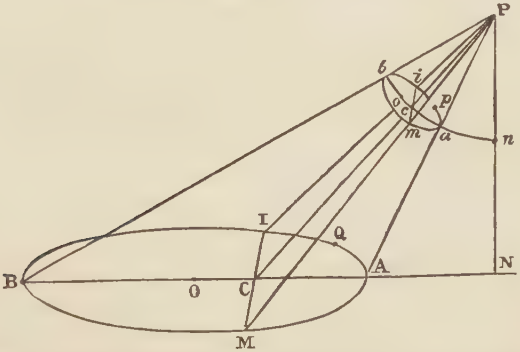
It is not easy to draw these equipotential curves independently, or even to deduce their typical equation from that of the lines of force by the mathematics of orthogonal trajectories.

The magnitude of the conical angle subtended at any point by a given circle can be expressed in finite terms by means of complete elliptic integrals of the third kind. The parameter involved in these integrals will depend on the way in which they are taken.

If a sphere of unit radius is described round P as centre, and lines are drawn from P to the points on the circum-

ference of the given circle, BMAI, fig. 4, these lines will intercept on the sphere a spherical ellipse, *bmai*, whose area is the conical angle subtended by the circle at P. The minor axis of this ellipse is the great circular arc *ab* determined by the lines PA, PB, while the major axis, *mi*, is determined by the chord, MI, of the circle which subtends a maximum angle, MPI, at P. This line is determined by drawing the bisector,

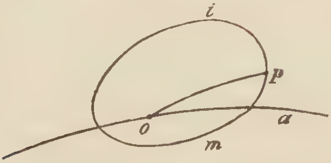
Fig. 4.



PC, of the angle BPA, meeting BA in C; then MI is the chord through C perpendicular to the plane BAP. The point c in which PC meets the surface of the sphere is the centre of the spherical ellipse.

Now, given any curve, *mpi*, fig. 5, on a sphere of unit

Fig. 5.



radius, its area is $\int(1-\cos\theta)d\phi$, where, if o is any point on the sphere *inside* the area, θ is the circular measure of the spherical radius vector op drawn to any point, p, of the curve, and ϕ is the angle between the radius op and any fixed arc, oa, drawn at o. If, as said, the pole o is inside the area, ϕ goes from o to 2π; but if o is *outside* the curve, the area has a different expression, viz.:

$$\int_0^o \cos\theta d\phi,$$

the longitude angle ϕ obviously starting and ending with a zero value. If o is *on* the curve, the expression for the area is again different.

In calculating the area of the above ellipse it would be natural to choose for pole (o) the point n in which the sphere is cut by the line PN; but this leads to difficulties when the position of P is such that n falls on the ellipse. This will happen when P is on any perpendicular to the plane of the circle of the current drawn at any point on its circumference; and, moreover, the choosing of n for pole will lead to expressions for the conical angle which present its values in forms which are apparently discontinuous for points P which project inside and outside the area of the given circle BMAI. Such discontinuity must not exist, and to get rid of it from the expressions requires troublesome transformations of elliptic integrals of the third kind.

We must, then, choose for pole a point which is always inside the spherical ellipse. The simplest point is the point o (fig. 4), in which the sphere is cut by the line PO which joins P to the centre, O, of the given circle. This point is, of course, always inside the ellipse.

Let, then, Q be any point on the given circuit, and p the point in which PQ cuts the ellipse. Taking for the fixed plane of longitude through o the plane baP, or BAP, and denoting the angle poa by ϕ, the area of the ellipse is

$$\int_0^{2\pi}(1-\cos op)d\phi, \text{ i. e., } 2\pi-\int_0^{2\pi}\cos op\,.\,d\phi.$$

Denoting, as before, the position of Q by the angle ψ, or QOA, we easily find, if PN$=z$, PO$=r$, ON$=x$,

$$d\phi=\frac{rz}{z^2+x^2\sin^2\psi}\,.\,d\psi,$$

$$\cos op=\frac{r^2-ax\cos\psi}{r\sqrt{r^2+a^2-2ax\cos\psi}},$$

Hence

$$\int_0^{2\pi}\cos op\,.\,d\phi=2z\int_0^{\pi}\frac{r^2-ax\cos\psi}{\sqrt{r^2+a^2-2ax\cos\psi}}\,.\,\frac{d\psi}{z^2+x^2\sin^2\psi}.$$

Putting $\psi = \pi - \chi$, this becomes

$$2z\int_0^{\pi}\left\{\rho\sqrt{1-k^2\sin^2\frac{\chi}{2}}+\frac{r^2-a^2}{\rho\sqrt{1-k^2\sin^2\frac{\chi}{2}}}\right\}\frac{d\chi}{z^2+x^2\sin^2\chi},$$

where, as before, $\rho = \text{PB}$, $\rho' = \text{PA}$, and $k^2 = 1 - \frac{\rho'^2}{\rho^2}$. If we put $\chi = 2\omega$, this becomes

$$z\int_0^{\frac{\pi}{2}}\left(\rho\Delta+\frac{r^2-a^2}{\rho\Delta}\right)\frac{d\omega}{z^2+x^2\sin^2 2\omega},$$

where $\Delta \equiv \sqrt{1-k^2\sin^2\omega}$.

To reduce this to elliptic integrals, we must resolve the fraction $1/z^2+x^2\sin^2 2\omega$ into two fractions. It is easily found that

$$z^2+x^2\sin^2 2\omega = z^2+4x^2(\sin^2\omega-\sin^4\omega)$$

$$=(\sqrt{z^2+x^2}+x-2x\sin^2\omega)(\sqrt{z^2+x^2}-x+2x\sin^2\omega).$$

Let ν denote the sine of the angle between PO and the axis of the current (or PN); then the expression, after resolution into partial fractions, becomes

$$\frac{z}{2r^2}\int_0^{\frac{\pi}{2}}\left(\rho\Delta+\frac{r^2-a^2}{\rho\Delta}\right)\left(\frac{1}{1-\nu+2\nu\sin^2\omega}+\frac{1}{1+\nu-2\nu\sin^2\omega}\right)d\omega.$$

The portion of this expression which has Δ in the denominator is at once the sum of two complete elliptic integrals of the third kind; and the portion which has Δ in the numerator is easily reduced to the same form. The result is

$$\frac{z}{r\rho}\int_0^{\frac{\pi}{2}}\left(\frac{r+a}{\text{N}\Delta}+\frac{r-a}{\text{N}'\Delta}\right)d\omega,$$

where $\text{N} \equiv 1-\nu+2\nu\sin^2\omega$, and N' is the value of N when ν is changed to $-\nu$. Hence we have two elliptic integrals of the third kind, one with $\left(\frac{2\nu}{1-\nu}, k\right)$ for parameter and modulus, and the other with $\left(\frac{-2\nu}{1+\nu}, k\right)$. In the usual notation,

then, we have for the complete expression of the conical angle subtended by the circuit at P the value

$$2\pi - \frac{2z}{\rho r}\left\{\frac{r+a}{1-\nu}\Pi\left(\frac{2\nu}{1-\nu}, k\right) + \frac{r-a}{1+\nu}\Pi\left(\frac{-2\nu}{1+\nu}, k\right)\right\}.$$

Of course it is not pretended that this expression is the most convenient for the purposes of calculation: the approximate value of the conical angle which is given by a series of spherical harmonics is that which should be employed; but it may be well to give the *complete* expression in the above form, which I have not seen published anywhere.

Discussion.

Prof. Perry thought the problem had been very prettily worked out, and hoped Prof. Minchin would be able to extend the solution to cylindrical coils,—a subject in which he (Prof. Perry) had long been interested.

Mr. Blakesley said Mr. Niven had pointed out to him that the locus of points for which the ratio ρ'/ρ is constant was an anchor ring.

Prof. Minchin maintained that the locus was a sphere, and on this subject a short discussion arose, in which the President, Prof. Perry, Prof. Minchin, Dr. Burton, and Prof. S. P. Thompson took part.

Dr. Sumpner described a method of drawing the lines of force of a circular current, or of any circuit, symmetrical about an axis, by a purely experimental and graphical process. The strengths of field were first determined at several points in the plane of the current, and from these results the points through which lines of force were to pass in order that their distribution might indicate the strength of the field at all points were graphically deduced.

XIV. *On the Differential Equation of Electrical Flow.*

By T. H. Blakesley, *M.A.**

The object of this paper is to point out that the theory of electrical discharge, as exemplified in the mathematical expressions employed to represent the physical facts, is incompetent to explain all the phenomena observed in extreme cases; and to show that the admission of certain properties of matter not usually recognized is the only way of satisfactorily obviating the imperfection of the existing views.

In some of the investigations I shall not employ exclusively algebraical symbolic methods, but, where it may more advantageously be adopted, I shall avail myself of the geometrical method. Such cases most frequently arise where magnitudes under consideration are capable of having negative values. All tidal effects, using the word in its most general sense, involve such magnitudes.

Electrical currents in a given conductor may have all possible values in one direction or in the opposite direction, but are otherwise restricted.

The projection of the line joining two points in space upon a fixed straight line is a geometrical magnitude of this sort. With respect to the direction in space, sometimes one of the projected points will be on one side of the second projected point, sometimes on the other. So that such a line has all the properties necessary for representing another magnitude of the same character.

In this way I shall most generally make the projection represent Electromotive Force, but occasionally Field of Magnetism at a point. As to matter of nomenclature, the only scientific term which I shall employ admitting of any doubtful interpretation, is the Effective Electromotive Force. By this term I intend to convey the idea of that electromotive force which is numerically equal to the product of the current and the resistance, at a point of time. As a department of State has recently employed the term in a

* Read March 24, 1893.

totally different sense, this statement has appeared to me to be necessary in the interests of proper explanation. The effective electromotive force is the algebraical sum of all the impressed and induced electromotive forces, and is here represented by E. If V is the sum of all the impressed electromotive forces and F is the sum of all induced electromotive forces, then the equation among these quantities is $V+F=E$ universally.

Geometrically, if A B, B C are lines whose projections on some one fixed straight line represent the sum of the impressed and the sum of the induced electromotive forces respectively, then the projection of A C will represent the effective electromotive force.

The three lines must form the sides of a triangle, those corresponding to the impressed and induced electromotive forces being taken the same way round the triangle, that corresponding to E being taken in the opposite direction.

Now if the actual changes in the magnitudes are harmonic, and of the same period, it is clear that the lines A B, B C, A C must remain of constant length and the triangle must rotate in its own plane at a uniform rate of such a value as to perform a complete revolution in the period of the harmonic change. The triangle thus shows admirably the way in which these magnitudes succeed one another in phase. It also follows from the properties of harmonic motion that if two magnitudes have the same harmonic period, but differ in phases by a quarter of the whole period, the corresponding lines to be projected are at right angles to each other. And hence the rate of variation of an harmonic magnitude differs in phase from the magnitude itself by a quarter of the period. But in the simplest case of a circuit being plied with an harmonic electromotive force V, it is generally considered that the induced electromotive force varies as the rate of change of the current; that is

$$F=-L\frac{dC}{dt}=-\frac{L}{R}\frac{dE}{dt}, \text{ for } E=RC,$$

where C is the current, R the resistance, and L is the coefficient of self-induction.

The equation already given then becomes

$$V - L\frac{dC}{dt} = E = RC.$$

Multiplying through by C and integrating through a complete period,

$$\int VC\,dt - L\int C\frac{dC}{dt}\,dt = R\int C^2 dt.$$

The first term represents the work done by the source of the disturbance.

The second term vanishes.

The third term represents work done in heating the circuit.

Hence the whole work done has gone to heat the circuit.

Now it is admitted on all hands that when the period is sufficiently short a radiation of energy into space takes place. A portion of this radiated energy is sometimes caught by means of a neighbouring circuit and converted into heat.

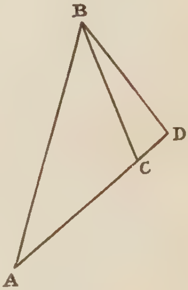
A coefficient of mutual induction and a corresponding extra term is then introduced into the equation. But are we to suppose that radiation would not proceed into space were there no neighbouring conductor? It is against probability, against the electromagnetic theory of light.

If electromagnetic waves are capable of being sent into space, we can no longer look upon the operation of establishing a current in a circuit as analogous to bending a stiff spring or displacing rigid wheelwork. The wheelwork must have indiarubber spokes or teeth.

The above equation takes no account of this radiation which is expended outside the wire, nor of any other work done elsewhere than in the conductor ; and this latter the equation states to be exactly equal to the energy expended in propagating the electromotive force. Even supposing a portion of the field is occupied by some material whose passage through a cycle of magnetization involves the loss of energy, in the form of heat, this, equally with wave-propagation through space occupied by perfectly elastic matter, will not be accounted for by the equation.

Now of such phenomena as radiation of energy in electro-magnetic waves, or absorption of energy in the field, there is ample evidence. Therefore an equation will not meet such cases in which the induced electromotive force is taken as entirely in quadrature with the current, or when F is wholly of the form $-L\frac{dC}{dt}$.

Hence, in the geometrical representation it is clear that the induced electromotive-force line must not be exactly at right angles to that of the effective electromotive-force line; *i. e.* the angle B C A is not exactly a right angle; and it is easy to see that it must be greater than a right angle, for B C may be resolved into B D . D C, where B D C is a right angle and A C D is one straight line. For then the whole work done is equal to $\frac{AD.AC}{2R}$. The work done in heating the conductor is $\frac{AC.AC}{2R}$, and the difference, or the work done in the field, is $\frac{AC.DC}{2R}$.



Hence, if D lies on the side of C nearer to A, A D would be less than A C, and the work done by the discharge would be less than that required to heat the conductor: in other words, energy would have to be received from space.

Hence the induced electromotive forces may be represented by two components—one B D in quadrature with the current, and one D C in opposition to it,

$$-L\frac{dC}{dt}-\lambda C,$$

where λ may or may not be a constant, but is in kind a resistance.

The equation among the electromotive forces may be written

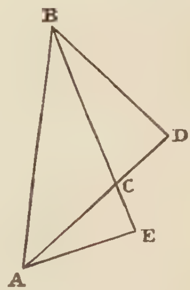
$$V-L\frac{dC}{dt}-\lambda C=RC.$$

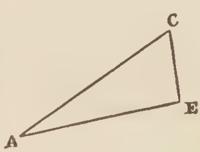
Multiplying through by $\mathrm{C}dt$ and integrating through a complete period,

$$\int \mathrm{VC}dt - \mathrm{L}\int \mathrm{C}\frac{d\mathrm{C}}{dt}dt = \mathrm{R}\int \mathrm{C}^2 dt + \int \lambda \mathrm{C}^2 dt.$$

The second term on the left vanishes as before, the first term representing the whole work done. On the right the first term heats the conductor and the second term gives energy to space.

We may go somewhat further into the causes of such an induced electromotive-force component if we employ the geometrical mode of symbolizing the electric quantities. BC, the induced electromotive-force line, should be at right angles to the induction through the circuit, for it is the rate of increase of the latter which produces the former. Hence if A E is a perpendicular let fall upon B C produced, A E will represent the phase of the magnetic induction. But A C being in phase with the current is in phase with the field. Hence E A C, or C B D which is equal to it, is a magnetic phase-lag, and A E may be said to be in phase with the *effective* field, and therefore with the induction. This suggests that if we employ the lower lines of the figure to represent *fields*, we may make up a triangle A C E such that A C is the impressed field, C E an induced field, and A E an effective field, of course when, as usual, projected on a fixed line ; C E being perhaps, though by no means certainly, at right angles to A E. However, whether C E here has in any case two components perpendicular and parallel respectively to A E or not, it appears very certain that the perpendicular component must exist. Assuming at first that it alone exists,—





If we employ small letters :—

v for impressed field $=\mathrm{AC}$,
f for induced field $=\mathrm{CE}$,
e for effective field $=\mathrm{AE}$,

l = coefficient of magnetic self-induction, so that

$$f=-l\frac{d\mathrm{I}}{dt};$$

and μ for the permeability, I for the rate of magnetic induction, *i. e.* per square centim., we have

$$v-l\frac{d\mathrm{I}}{dt}=e=\frac{\mathrm{I}}{\mu}.$$

To obtain an equation of energy from this we must multiply (not by I, as analogy would at first sight perhaps dictate) by $\frac{d\mathrm{I}}{dt}\cdot dt\times$ cross section, for the formula for energy is

$$[l^2t^{-2}m]\equiv[\mu^{-\frac{1}{2}}l^{-\frac{1}{2}}t^{-1}m^{\frac{1}{2}}][\mu^{\frac{1}{2}}l^{\frac{5}{2}}t^{-2}m^{\frac{1}{2}}]t$$

$$\equiv\Big[\mathrm{Field}\Big]\Big[\frac{\mathrm{lines}}{t}\Big]t$$

$$\int v\frac{d\mathrm{I}}{dt}\cdot dt-l\int\left(\frac{d\mathrm{I}}{dt}\right)^2dt=\int\frac{\mathrm{I}}{\mu}\frac{d\mathrm{I}}{dt}\cdot dt.$$

Here the term on the right hand disappears necessarily, and the work expended, if any, is equal to $l\int\left(\frac{d\mathrm{I}}{dt}\right)^2dt$. Hence this work vanishes only if $l=0$, *i. e.*, if there is no component field in quadrature with the induction ; a curious antithesis to the electric problem. If there were a field induced in phase with the induction, it would not result in the dissipation of energy. No argument for such a state of things can be drawn on the score of loss of energy.

If the phases of magnetism in any cycle coincided with the phases of field, there could be no such thing as hysteresis ; and, further, no radiation of energy from an alternating magnet.

But both these phenomena have been for many years recognized. It follows, therefore, that when induction through any space changes, a magnetic field is induced acting counter to the change.

Now it may be noticed that the tangents of the angles of lag, whether electric or magnetic, involve a coefficient in their numerator and the value of the period in their denomi-

nator. They therefore become larger as the period is made less. It might therefore happen that extreme rapidity of change would be necessary before a lag of current or of induction could be detected. Electric lags, or, at all events, the coefficients of self-induction can readily be measured. Magnetic lags have been measured by the author in certain cases by special artifices, but when we deal with a medium of small permeability, as air, the period must be extremely minute to make the lag-angle sensible. and as yet no machines possess a sufficient frequency to effect it. Recourse has been had to the rapid oscillations which take place when a Leyden jar is discharged. In these cases the radiation has been frequently caught and approximately measured, and it is therefore in these very cases that the rectification of the formula becomes important.

I propose to investigate by geometry and otherwise the conditions under which a Leyden jar is discharged. Geometry especially will afford an excellent and graphic insight into the question of the oscillatory discharge.

As in the case of the sustained discharge, I shall first take the usual formula, obtain the geometrical illustration of it, and, observing where the defect shows itself, pass by an easy transition to the truer state of things.

In the case of the discharge of a condenser through a simple circuit removed from proximity with any other, we have the following equations to deal with :—

The general equation $V+F=E$, ... (i.)

$$E = RC, \quad . \quad . \quad . \quad . \quad . \quad . \quad (ii.)$$

$$F = -L\frac{dC}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (iii.)$$

$$C = -K\frac{dV}{dt}, \quad . \quad . \quad . \quad . \quad . \quad (iv.)$$

where

C is the current discharging the condenser ;
K is the capacity of the condenser ;
L is the coefficient of self-induction ;
R is the resistance ;
F is the induced electromotive force ;

E is the effective electromotive force ;
V is the potential difference of the plates of the condenser.

Eliminating E and F from (i.) by means of (ii.) and (iii.),

$$V-L\frac{dC}{dt}-RC=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (v.)$$

Differentiating,

$$\frac{dV}{dt}-L\frac{d^2C}{dt^2}-R\frac{dC}{dt}=0,$$

and substituting for $\frac{dV}{dt}$ from (iv.),

$$-\frac{C}{K}-L\frac{d^2C}{dt^2}-R\frac{dC}{dt}=0,$$

or

$$C+KR\frac{dC}{dt}+LK\frac{d^2C}{dt^2}=0,$$

the differential equation of C.

Since $C=\frac{E}{R}$, it follows that the equation

$$E+KR\frac{dE}{dt}+LK\frac{d^2E}{dt^2}=0$$

is also true, and therefore is the differential equation of E.

Again, from (v.) and (iv.), since $\frac{dC}{dt}=-K\frac{d^2V}{dt^2}$, obtained by differentiating (iv.),

$$V+RK\frac{dV}{dt}+LK\frac{d^2V}{dt^2}=0.$$

This is the differential equation of V.

Thirdly, differentiating twice the equation

$$V+F-RC=0,$$

$$\frac{d^2V}{dt^2}+\frac{d^2F}{dt^2}-R\frac{d^2C}{dt^2}=0,$$

from (iii.) by differentiation it is seen that

$$\frac{d^2C}{dt^2}=-\frac{1}{L}\frac{dF}{dt},$$

and from (iv.),

$$\frac{d^2V}{dt^2} = -\frac{1}{K}\frac{dC}{dt},$$

which is further reduced to $\frac{1}{K}\frac{F}{L}$ by (iii.).

Hence $$\frac{F}{KL} + \frac{d^2F}{dt^2} + \frac{R}{L}\frac{dF}{dt} = 0,$$

or

$$F + KR\frac{dF}{dt} + KL\frac{d^2F}{dt^2} = 0,$$

the differential equation of F.

It is thus clear that the variables V, F, E, C all have the same form of differential equation, viz. :—

$$E + KR\frac{dE}{dt} + KL\frac{d^2E}{dt^2} = 0.$$

Of course, to make this equation homogeneous,

KR is of the order (time) ;
KL ,, ,, (time)^2.

KL may be written $KR \cdot \frac{L}{R}$, or still better $\frac{KR}{2} \cdot \frac{2L}{R}$.

If we write $$\frac{2L}{R} = t_1$$

and $$\frac{KR}{2} = t_3,$$

t_1 and t_3 are time-constants of the circuit, and the differential equation may be written

$$E + 2t_3\frac{dE}{dt} + t_1 t_3\frac{d^2E}{dt^2} = 0.$$

In this expression the two time-constants may be considered to be independent.

To obtain a geometrical representation of the changes :—

(1) Suppose a line, whose length is r, to shrink logarithmically so that its change is represented by the equation

$$\frac{dr}{dt} = -\frac{r}{t_1},$$

where t_1 is a time-constant.

Then
$$r = ae^{-\frac{t}{t_1}},$$

where a is the value of r at the beginning of the time, and t_1 appears as the time taken for r to shrink to $\frac{1}{e}$ of itself.

(2) Secondly, suppose a straight line in a plane to constantly change in *direction* at a uniform rate, in the same sense. If θ is the angle measured from a fixed direction,

$$\frac{d\theta}{dt} = \frac{2\pi}{t_2},$$

where t_2 is a time-constant. Hence

$$\theta = \frac{2\pi}{t_2} \cdot t.$$

Whence t_2 appears as the time required to describe 2π.

(3) Suppose a line to undergo both the changes contemplated, which is possible, since one is a change of length, the other a change in direction. Then, eliminating the time, we have

$$r = ae^{-\frac{\theta}{\frac{2\pi t_1}{t}}},$$

or $r = ae^{-\frac{\theta}{\tan\beta}}$, where $\tan\beta = \frac{2\pi t_1}{t_2}$.

This is the equation of the equiangular spiral, with the characteristic angle β, whose value merely depends upon the two time-constants t_1 and t_2.

(4) Now imagine this length r constantly projected on some fixed straight line, and for simplicity take this straight line as at right angles to the direction in which $\theta = 0$.

Then the projection under consideration (E) has for its expression

$$\mathrm{E} = a \cdot e^{-\frac{\theta}{\tan\beta}} \sin\theta;$$

i. e., it consists of a constant factor, a logarithmical factor, and a rhythmical or harmonic factor.

Substituting for θ its value $\frac{2\pi}{t_2} . t$,

$$\mathrm{E} = a . e^{-\frac{2\pi}{t \tan\beta} . t} \sin\frac{2\pi}{t_2} . t,$$

$$\mathrm{E} = a e^{-\frac{\theta}{\tan\beta}} \sin\theta,$$

$$\frac{d\mathrm{E}}{d\theta} = a e^{-\frac{\theta}{\tan\beta}} \cos\theta - \frac{a}{\tan\beta} e^{-\frac{\theta}{\tan\beta}} \sin\theta$$

$$= a e^{-\frac{\theta}{\tan\beta}} \cos\theta - \frac{\mathrm{E}}{\tan\beta}.$$

$$\frac{d^2\mathrm{E}}{d\theta^2} = -\frac{1}{\tan\beta}\frac{d\mathrm{E}}{d\theta} - a e^{-\frac{\theta}{\tan\beta}} \sin\theta - \frac{a}{\tan\beta} e^{-\frac{\theta}{\tan\beta}} \cos\theta$$

$$= -\frac{1}{\tan\beta}\frac{d\mathrm{E}}{d\theta} - \mathrm{E} - \frac{1}{\tan\beta}\left\{\frac{d\mathrm{E}}{d\theta} + \frac{\mathrm{E}}{\tan\beta}\right\}$$

$$= -\frac{2}{\tan\beta}\frac{d\mathrm{E}}{d\theta} - \mathrm{E}\left\{1 + \frac{1}{\tan^2\beta}\right\};$$

therefore

$$\mathrm{E} + \frac{2\tan\beta}{1+\tan^2\beta}\frac{d\mathrm{E}}{d\theta} + \frac{\tan^2\beta}{1+\tan^2\beta}\frac{d^2\mathrm{E}}{d\theta^2} = 0.$$

Now

$$\frac{d\mathrm{E}}{d\theta} = \frac{d\mathrm{E}}{dt} . \frac{dt}{d\theta} = \frac{d\mathrm{E}}{dt} . \frac{t_2}{2\pi},$$

therefore

$$\frac{d^2\mathrm{E}}{d\theta^2} = \frac{d^2\mathrm{E}}{dt^2} . \frac{dt}{d\theta}\frac{t_2}{2\pi} = \frac{d^2\mathrm{E}}{dt^2}\left(\frac{t_2}{2\pi}\right)^2,$$

therefore

$$\mathrm{E} + \frac{2\tan\beta}{1+\tan^2\beta}\frac{t_2}{2\pi} . \frac{d\mathrm{E}}{dt} + \frac{\tan^2\beta}{1+\tan^2\beta}\left(\frac{t_2}{2\pi}\right)^2\frac{d^2\mathrm{E}}{dt^2} = 0;$$

which, since $\tan\beta = \frac{2\pi t_1}{t_2}$, becomes

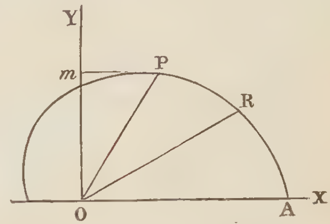
$$\mathrm{E} + \frac{2t_1}{(1+\tan^2\beta)}\frac{d\mathrm{E}}{dt} + \frac{t_1^2}{(1+\tan^2\beta)}\frac{d^2\mathrm{E}}{dt^2} = 0;$$

and if t_3 is written for $\frac{t_1}{(1+\tan^2\beta)}$,

$$\mathrm{E} + 2t_3\frac{d\mathrm{E}}{dt} + t_1 t_3\frac{d^2\mathrm{E}}{dt^2} = 0.$$

An equation which is at once comparable with those obtained in the Electrical problem.

It thus appears that the variables in the problem of electrical discharge under consideration may be represented by the projections of three sides of a triangle, which is constantly undergoing uniform rotation and linear logarithmic shrinking. Let the figure represent a portion of the appropriate curve whose characteristic angle is β, and let O R be some radius vector. Then the projection of O R on O Y will be a maximum when the tangent at R is parallel to O X. Let P be such a position, and let P M be the tangent at P. Then M P O = P O X = β.

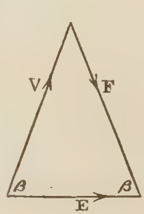


Now suppose O A the line representing (in its projection) the effective electromotive force about to change sign through the value zero. This means that the current is about to change sign, and the condenser having been receiving current is about to begin to be discharged, *i. e.* its charge and therefore potential difference is a maximum. Then the line of P.D. must make the angle P O A with the line of *effective E.M.F.*

Again, the E.M.F. of self-induction is zero when the current is at a maximum, by the nature of the ordinary hypothesis.

Therefore, when the line representing E makes an angle β with O X, the line representing induced E.M.F. (F) must be parallel with it. Hence it also makes an angle β with the line of effective E.M.F.; but in phase lags behind it, whereas the P.D. is in advance by that angle.

Thus, if on any line taken as base we construct an *isosceles* triangle of appropriate base angles, the sides will represent the P.D. of the condenser and the induced E.M.F. of self-induction respectively, and the base will represent the effective E.M.F. It only remains to rotate the triangle with appropriate speed and to allow it to shrink at the due logarithmic rate.



The properties of the triangle agree exactly with the electrical properties.

The angle β is such that $\tan\beta = \frac{2\pi t_1}{t_2}$;

therefore
$$\cos^2\beta = \frac{1}{1+\left(\frac{2\pi t_1}{t_2}\right)^2} = \frac{t_3}{t_1} = \frac{\frac{KR}{2}}{\frac{2L}{R}} = \frac{KR^2}{4L};$$

therefore $\cos\beta = \sqrt{\frac{K}{L}}\cdot\frac{R}{2}$, and β can have a real existence when $\frac{KR^2}{4L} < 1$;—the condition of Oscillatory Discharge.

The complete period is t_2, and is obtained in the electrical quantities thus :—

$$\tan\beta = \frac{2\pi t_1}{t_2};$$

therefore

$$t_2^2 = \frac{(2\pi)^2 t_1^2}{\tan^2\beta} = \frac{(2\pi)^2 t_1^2}{\sec^2\beta - 1};$$

and $\cos^2\beta = \frac{t_3}{t_1}$ already obtained ;

therefore
$$t_2^2 = \frac{4\pi^2 t_3 t_1}{1-\frac{t_3}{t_1}} = \frac{4\pi^2 KL}{1-\left(\frac{KR^2}{4L}\right)};$$

therefore

$$t_2 = \frac{2\pi\sqrt{KL}}{\left(1-\left(\frac{KR^2}{4L}\right)\right)^{\frac{1}{2}}},$$

the form usually quoted if we neglect the second term of the denominator.

I purpose to show that in a discharge of the sort here contemplated (which has been shown to be the result of the ordinary premisses given at page 222) there will be no work done by any electromotive force which lags at an angle β behind the current, provided the initial condition is one of zero-current. And, further, that the source of E.M.F., which is represented by the side of the isosceles triangle in advance by the angle β, of the effective E.M.F., does all the work of

heating the circuit and no more. It will thus be seen that there is *no provision in the theory for expenditure of power in the field*, and hence that the theory does not explain the well recognized phenomenon of radiation into space.

To establish the above-mentioned propositions, take the product of the projections of two lines undergoing variations corresponding to the two radii vectores of two equiangular spirals of the same characteristic angle β and period, and differing in phase by the angle 2γ.

One of these quantities may be expressed by

$$a e^{-\frac{\theta+\gamma}{\tan\beta}} \sin \overline{\theta+\gamma}.$$

The other by

$$b e^{-\frac{\overline{\theta-\gamma}}{\tan\beta}} \sin \overline{\theta-\gamma}.$$

The product is

$$ab \,.\, e^{-\frac{2\theta}{\tan\beta}} \sin \overline{\theta+\gamma} \sin \overline{\theta-\gamma},$$

or

$$ab \,.\, e^{-\frac{2\theta}{\tan\beta}} (\sin^2\theta - \sin^2\gamma).$$

This quantity, multiplied into an element of time dt, has to be integrated through one period. Since $\frac{d\theta}{dt} = \frac{2\pi}{t_2}$, the integral I becomes

$$\frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} (\cos 2\gamma - \cos 2\theta)\, d\theta,$$

or

$$\frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} \cos 2\gamma\, d\theta - \frac{abt_2}{4\pi}\int e^{-\frac{2\theta}{\tan\beta}} \cos 2\theta\, d\theta.$$

The first term of the integral is

$$-\frac{abt_2}{8\pi} \cos 2\gamma \tan\beta\, e^{-\frac{2\theta}{\tan\beta}}.$$

The second term is

$$\frac{abt_2}{8\pi} \sin\beta \cos \overline{\beta + 2\theta}\, e^{-\frac{2\theta}{\tan\beta}};$$

and therefore the integral is expressed :—

$$\mathrm{I} = \frac{abt_2}{8\pi} e^{-\frac{2\theta}{\tan\beta}} \{\sin\beta \cos \overline{\beta + 2\theta} - \tan\beta \cos 2\gamma\}.$$

This expression has to be taken between limits. If we contemplate one revolution only the limits will be $\Theta_1 + 2\pi$ and θ_1, and the Definite Integral becomes

$$\frac{abt_2}{8\pi}\{-\sin\beta\cos\overline{\beta+2\theta_1}+\tan\beta\cos 2\gamma\}e^{-\frac{2\theta_1}{\tan\beta}}\left(1-e^{-\frac{4\pi}{\tan\beta}}\right).$$

If the limits are infinity and θ_1 the integral becomes

$$\frac{abt_2}{8\pi}\{\tan\beta\cos 2\gamma-\sin\beta\cos\overline{\beta+2\theta_1}\}e^{-\frac{2\theta_1}{\tan\beta}}. \quad . \quad (\alpha)$$

Either of these expressions becomes zero when

$$\tan\beta\cos 2\gamma=\sin\beta\cos\overline{\beta+2\theta_1}$$

or

$$\cos 2\gamma=\cos\beta\cos\overline{\beta+2\theta_1},$$

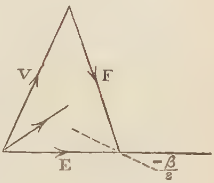
showing that the condition that no work shall be done in the electric problem depends on the initial circumstances, *i.e.* θ_1 is involved. If $2\gamma=\beta$ the condition of no work becomes

$$\cos\overline{\beta+2\theta_1}=1,$$

which is satisfied when

$$\theta_1=-\frac{\beta}{2}.$$

Hence if the initial condition be that of no current, the line bisecting the angle between the line of effective E.M.F. and that of the self-induced E.M.F. makes $-\frac{\beta}{2}$ with the line $\theta=0$, and it is thus proved that on the whole no work is done in the field.



If, on the other hand, we make

$$\theta_1=+\frac{\beta}{2},$$

and start from a point where the current is zero, we have in the above expression, when proper substitutions are made for a and b, the value of the work done on the circuit by the discharging condenser. The integral between infinity and θ_1 becomes, when $\theta_1=\frac{\beta}{2}$ and $2\gamma=\beta$,

$$\frac{ab\,.\,t_2}{8\pi}\{1-\cos 2\beta\}\sin\beta\,.\,e^{-\frac{\beta}{\tan\beta}},$$

or

$$\frac{ab \cdot t_2}{4\pi} \sin^3\beta\, e^{-\frac{\beta}{\tan\beta}}.$$

In this case $b = \frac{E}{R}$ and $ae^{-\frac{\beta}{\tan\beta}} \sin\beta$ is the potential difference between the plates of the condenser at starting $= V_1$, say, $= V \sin\beta$. Hence the expression becomes

$$\frac{E}{R} \cdot \frac{t_2}{4\pi} \sin^3\beta \cdot V$$

(E and V being now the full sides of the triangle, properly interpreted), and $\frac{E}{2 \cdot V} = \cos\beta$ by the geometry of the triangle; and further,

$$\tan\beta = \frac{2\pi t_1}{t_2} \text{ and therefore } \frac{t_2}{4\pi} = \frac{t_1}{2\tan\beta},$$

and the work

$$= \frac{E^2}{R} \frac{1}{2\cos\beta} \frac{t_1}{2\tan\beta} \sin^3\beta, \text{ or } \frac{E^2}{R} \frac{t_1 \sin^2\beta}{4},$$

which is the expression we should obtain if we integrate the square of the current multiplied by Rdt, seen as follows:—

In the general expression (α) obtained above for the product of the projections make $a = E$, $b = \frac{E}{R}$, and $\theta_1 = 0$, $\gamma = 0$, the expression (α) becomes

$$\frac{E^2 t_2}{8\pi R} \{\tan\beta - \sin\beta\cos\beta\},$$

or

$$\frac{E^2}{4R} \cdot \frac{t_1}{\tan\beta} \{\tan\beta - \sin\beta\cos\beta\},$$

or

$$\frac{E^2}{4R} t_1 \sin^2\beta, \text{ as above.}$$

Thus the whole of the work goes to heat the wire, and, further, substituting in the equation for E in terms of V, it may be shown to be entirely derived from the charged condenser.

The work may be written, eliminating E,

$$\frac{2V^2\cos\beta}{R} \frac{t_2}{4\pi} \sin^3\beta,$$

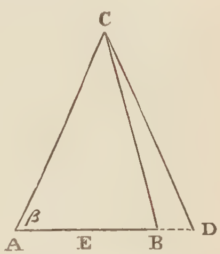
or
$$\frac{V^2}{R}\frac{t_1}{\tan\beta}\cos\beta\sin^3\beta.$$

Now $V_1^2 = V^2\sin^2\beta$, and thus the work is
$$V_1^2\frac{t_1}{R}\cos^2\beta,$$

or, since $\cos^2\beta = \frac{t_3}{t_1}$,
$$= \frac{V_1^2 t_3}{R} \text{ and } t_3 = \frac{KR}{2}$$
$$= \frac{V_1^2 K}{2},$$

which is the ordinary expression for the energy stored in the condenser; and this appears from the investigation to be entirely expended in heating the circuit, and there is no margin for the exhibition of power elsewhere.

Suppose a line A B to represent (E) the line of effective E.M.F. At the extremity A set off A C as the direction of the line representing the P.D. of the condenser.



Then, as the condenser contains all the energy that is going to be expended on the circuit and on the æther, from what has been said it is clear that A C must be rather longer than the side of the isosceles triangle; for, if not, the energy stored will not do more than heat the circuit. If, therefore, a perpendicular be dropped upon A B from C, it will fall at a point nearer B than A.

Join C B, and, further, draw C D to meet A B produced in D, and so that C D A is an isosceles triangle on A D as base, and therefore C D A $=\beta$. Now C B must be the line representing the resultant of the induced electromotive forces F, and however complicated the case may be this line C B is equivalent to two components C D, D B; of which C D results in no expenditure of power because it is in a phase β *behind* the current, and D B is in phase directly opposed to the current, and therefore resulting in whatever expenditure of energy takes place outside the circuit, and therefore in the æther or in magnetic bodies, or in neighbouring or surrounding conductors. As in the former case of sustained oscillations,

it may be shown that B C D is a *magnetic lag* necessary for the exhibition of such phenomena.

The electromotive force D B may be expressed by $-\lambda C$ as before, and the general equation

$$V+F=E$$

takes the form

$$V-L\frac{dC}{dt}-\lambda C=RC,$$

and, as this may be written

$$V-L\frac{dC}{dt}=(R+\lambda)C,$$

we see that the extra consideration required to express the actual state of things is simply that the resistance of the circuit is virtually increased. In the previous work it is necessary to write $(R+\lambda)$ in all the equations.

The actual work done altogether is derived from the charged condenser. This is divided between the circuit and the field in the ratio $R:\lambda$.

It may happen, therefore, that if the circumstances of the discharge are such as to make λ very large in comparison with R, the ordinary heating-effect may be minimized. Among such causes is frequency, and in this consideration is to be found the true explanation of some of the experiments of M. Nikola Tesla. The energy of the discharges which that physicist encountered was expended in chief part in radiation which his body did not check, and not in current through his body. It is here suggested that the best way to measure radiation would be to measure the *defect* in the heating of a circuit, taking care to note the P.D. of the condenser at the moment previous to discharge.

In ordinary *sustained* oscillations, as derived from a machine, the alternations are not of sufficient frequency to make the effect of λ perceptible. Electromotive forces of induction involve the period in their denominators, and it is reasonable to suppose that induced magnetic fields do the same; and if the period of the electromagnetic vibrations becomes comparable with that of light, it is conceivable that mere heating might vanish, as in the solar spectrum light has less heating-effect than radiation of smaller frequency from the same source.

Discussion.

Prof. Perry thought the C^2R term would not represent the heating of the wire when the oscillations were rapid, owing to the distribution of current not being uniform over the section of the conductor. Maxwell had shown that certain throttling terms had to be considered. In condenser discharges the complete equation would have many terms.

Prof. O. J. Lodge said the best definition of R in such cases was that derived from Joule's law rather than that of Ohm. Frequency was very important in the radiation of energy, but even at ordinary frequencies of alternators some energy was radiated. Referring to Tesla's experiments, he said the reason why no serious consequences followed was that there was not much energy behind them. High frequency might be instrumental in preventing injury, but this he thought remained to be proved.

Dr. Sumpner pointed out that losses other than C^2R (R being the ordinary resistance of the conductor) had to be taken into account. In some cases, such as transformers on open circuit, the effective resistance might be one thousand times that of the coil. To discuss completely the problem taken up by Mr. Blakesley, it would be necessary to take account of non-uniform distribution of current both across and along the conductor, as well as the character of the magnetic and electric fields surrounding the circuit.

Mr. Swinburne thought there was a tendency to overestimate the rate of high-frequency currents, for unless the coils of transformers were assumed geometrically coincident, calculations were difficult. Errors of hundreds per cent were quite possible. In Tesla's experiments no great power was involved, for the transformer could not give out any large power.

Mr. Blakesley, in reply, said the term R was such that C^2R represented the whole waste in the conductor, whilst $C^2\lambda$ included everything wasted outside the conductor.

XV. *Liquid Friction.* *By* JOHN PERRY, *F.R.S.*, *assisted by* J. GRAHAM, *B.A.*, *and* C. W. HEATH*.

[Plate II.]

A PIECE of apparatus such as is used in this investigation was designed and partly constructed in Japan in 1876 ; it is described in my book on Practical Mechanics (1883). The specimen actually used by us was constructed at the Finsbury Technical College in 1882, and has been occasionally used since that time, but no complete sets of observations were attempted till October 1891.

The simplest hydrodynamical condition of viscous fluid is that of the fluid bounded by two infinite parallel planes, the fluid in one boundary being at rest, the velocity in the other boundary being constant and in the plane. Motions in a pipe and near a vibrating disk, or even near a steadily rotating disk, are rather complicated. Our apparatus was designed to approach as nearly as possible to the conditions subsisting between the infinite planes. Between two such planes, if V is the constant velocity of one of them, the other being at rest, and b is their distance asunder, the fluid being of uniform density, and gravity being neglected, if z is measured at right angles from the fixed plane, the equation of steady motion is

$$\frac{d^2v}{dz^2} = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

or

$$v = \mathrm{V}z/b \, ; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and the tractive force per unit area required at the moving plane to maintain the motion is $\mu \mathrm{V}/b$, where μ is the coefficient of viscosity. We have used, instead of planes, concentric cylindric surfaces of as large radii and as small difference in radii as could be conveniently constructed and used (see Pl. II. fig. 2).

E E E is a cylindric trough, of which the curved parts E and E are brass. The inner and outer radii of this trough are 10·39 and 12·65 centimetres. C, which forms the bottom, is of iron ; and the whole trough can be rotated about its

* Read March 24, 1893.

vertical axis at any desired speed by driving the pulley P from a coned pulley D with numerous steps.

G is a hollow brass cylinder supported by a steel wire L, of 0·037 centim. diameter, 67·78 centim. long, whose axis coincides with the axis of the trough and the axis of rotation. G may be raised or lowered relatively to the trough. The outer radius of G is 11·63 centim., the inner being 11·41 centim. The whole apparatus is supported on a stand, with three adjustable feet. We exhibit also some photographs of the apparatus in position, showing how it was driven.

The trough contains the liquid whose viscosity is to be measured: when it rotates, G tends to rotate; and when for any constant speed G is in equilibrium, the twist in the steel wire measures the torque due to the tractive forces with which the liquid acts upon G at its inner and outer surfaces. The twist was measured by the angular motion of a pointer clamped on the wire at a distance of 59 centim. from the fixed end.

To test the accuracy of our assumption that the fluid behaved as if between parallel plane surfaces, let us consider the actual motion in which the stream-lines are circles. Consider the motion of a stream-tube of section $\delta r\,\delta x$, x being measured axially and r radially. The tangential force on unit cylindric surface of radius r is $\mu\left(\frac{dv}{dr}-\frac{v}{r}\right)$, if v is the velocity. The moment due to all such forces on the inner surface of our ring is

$$2\pi r^2 \,.\, \mu\left(\frac{dv}{dr}-\frac{v}{r}\right)\delta x.$$

The moment tending to increase the velocity of the ring due to forces on the cylindric parts of it is therefore

$$2\pi\mu \,.\, r^2\left(\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}\right)\delta r \,.\, \delta x\,;$$

also, due to the plane faces we have the moment

$$2\pi\mu r^2\frac{d^2v}{dx^2}\delta r \,.\, \delta x.$$

Equating the sum of these to the rate of increase of the moment of momentum of the ring, we have

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}+\frac{d^2v}{dx^2}=\frac{\rho}{u}\frac{dv}{dt} \quad . \quad . \quad . \quad . \quad (3)$$

as the equation of motion in co-axial circular stream-lines.

Now the discontinuity at the edge, and also the nearness of the bottom of the trough, cause the term $\frac{d^2v}{dx^2}$ to be important; but the solution seems to be very difficult. Maxwell satisfied himself (Collected Papers, vol. ii. pp. 16–18) that the discontinuity at the edge of a vibrating disk could be allowed for as a virtual increase in the radius of his disk, and the assumption that the behaviour of his fluid was the same as if his disk were part of an infinite disk. The correction not being readily obtained for a disk, he assumed it to be the same as for the straight edge of an infinite plane surface. We are certainly not less correct in taking the same correction for the edge of our cylinder *. Following Maxwell, therefore, we assumed that when our cylinder G was immersed to the depth AB or l in the fluid it was really a portion of length $l+\lambda$ of an infinite cylinder of the same diameter. We therefore neglect $\frac{d^2v}{dx^2}$ in (3), and we use

$$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{dv}{dr}-\frac{v}{r^2}=\frac{\rho}{\mu}\frac{dv}{dt}. \quad . \quad . \quad . \quad . \quad . \quad (4)$$

When the motion is steady, that is when $dv/dt=0$, the solution is

$$v = \mathrm{A}r+\mathrm{B}/r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

If $v=v_1$ when $r=\mathrm{R}_1$, and $v=0$ when $\mathrm{R}=\mathrm{R}_2$, then

$$v = v_1\mathrm{R}_1(r-\mathrm{R}_2^2/r)/(\mathrm{R}_1^2-\mathrm{R}_2^2).$$

We must now distinguish between the space outside the

* It is to be remarked that Maxwell assumed, generally, that there was no radial motion of his fluid. Now there must have been radial motion, his disks resembling centrifugal fans in their action, creating a variable flow always outwards between his fixed and moving disks; and the energy wasted in producing this flow is neglected by him. We do not know the amount of this error, and he may have satisfied himself as to its insignificance. Prof. Maurice FitzGerald in criticizing this proof has pointed out the fact that on James Thomson's theory of river bends there must exist a radial motion of an interesting kind in our apparatus.

suspended cylinder and the space inside it. The radii of the inner and outer surfaces of the suspended cylinder are 11·41 and 11·63 centim., and the inner and outer radii of the trough are 10·39 and 12·65 centim.

Our cylindric surfaces were not perfectly true, although great care was taken to make them so; and the radii given are only average dimensions. But, inasmuch as slightly tilting the apparatus or otherwise putting the axis of the suspended cylinder out of coincidence with the axis of the trough made only small differences in the observations, we did not think that such inaccuracies of workmanship or measurement as existed could affect our results.

Even when the tilting of the apparatus was quite evident to the eye, the tractive torque was found to be only slightly increased by the tilting. Of course, as the suspended cylinder got closer and closer to the side of the trough the torque did increase, and became very large when the suspended cylinder nearly touched the side of the trough.

Again, it was observed that at our highest speeds the amount of wetted surface did not perceptibly alter; and we are, we think, justified in assuming that the surface of the liquid was always a plane surface.

It is evident that the tractive forces on the suspended cylinder are the same whether we assume the trough to revolve steadily at ω radians per second, the suspended cylinder being at rest, or the suspended cylinder to revolve steadily at ω radians per second and the trough to remain at rest. We shall therefore, for ease of calculation, always assume the trough to be at rest and the suspended cylinder to be revolving at ω radians per second. Then the velocities of its inner and outer surfaces are, in centimetres per second, $11·41\,\omega$ and $11·63\,\omega$.

On any cylindric surface the tractive force per unit area being $\mu\left(\frac{dv}{dr}-\frac{v}{r}\right)$ is $-\frac{2B}{r^2}\mu$ from (5); so that, whether for the outer or inner space, if R_1 is the radius of the suspended moving cylindric surface, and R_2 the radius of the fixed surface, the tractive moment per centim. of length is

$$\pm 4\pi\omega\mu R_1^2/(R_1^2/R_2^2-1).$$

Taking actual sizes, this is 0·5 per cent. greater than the value obtained by calculating the forces on the assumption that the fluid moves in plane layers as in (2), b being the actual thickness of fluid 1·02 centim., and V being the actual velocity at the mean radius. We may, in fact, imagine the speeds to be increased by 0·5 per cent., and make all calculations as to viscosity on the assumption of motion in plane layers.

The tractive torque per centimetre of length of cylinder is, in our case, $19010\mu\omega$, or $1991n\mu$ if the angular velocity is given as n turns per minute. If l is the wetted length in centimetres, and λ is the virtual additional length representing the edge effect, the total torque is $1991n\mu(l+\lambda)$. The total observed motion of the pointer being D degrees, and the torque per degree being a, the torque due to tractive forces acting on the cylinder is

$$a\mathrm{D} = 1991\,n\mu(l+\lambda)\ ;$$

and if this law is found to be true experimentally, then

$$\mu = a\mathrm{D}/\{1991(l+\lambda)n\}. \quad . \quad . \quad . \quad . \quad (6)$$

Two methods of determining the torsional constant of the wire were employed :—

First Method.—A fine cotton thread was wound round the outside of the suspended cylinder and passed over a nearly frictionless pulley (the pulley of an Attwood's machine) to a scale-pan. The thread was nearly horizontal as it left the cylinder. In this way it was found that the twisting moment required to produce a pointer-rotation of one degree was 1531 dyne-centimetres. In making the measurement, as the weight of the scale-pan and its contents was gradually increased, the steel wire was drawn away from the vertical, and therefore from the middle of the scale; but the stand was tilted to counteract this effect.

The effects due to solid friction were eliminated by taking the mean of the limiting weights for equilibrium. When the weight was 30 grams, one tenth of a gram either added to or taken from the scale-pan produced a perceptible change in the position of the pointer ; so that the solid friction was small.

Second Method.—The suspended cylinder was allowed to vibrate, twisting and untwisting the wire; and its times of oscillation were noted. The observations were repeated when a known moment of inertia had been added. Unloaded, it made 40 complete oscillations in 583 seconds, or one oscillation in 14·575 seconds. We then attached to the cylinder an iron bar of rectangular section, whose own moment of inertia had been determined accurately by previous experiments (found to agree with calculation on the assumption that it was homogeneous), this moment of inertia being 566·2 (in gram-centimetre^2 units). The time of a complete oscillation was now found to be 21·425 seconds. It follows that the moment of inertia of the suspended cylinder is 487·72, and the torsional constant of the wire is readily obtained. This constant being corrected on account of the position of the pointer, it follows that to produce a rotation of the pointer of one degree requires a torque of 1552 dyne-centimetres. This is greater than the constant derived from direct measurement by $1\frac{1}{3}$ per cent.; but, on the whole, we are rather inclined to accept the number obtained directly, as we are not quite sure that the mean position of the iron bar was at right angles to the magnetic meridian.

Eight quite independent measurements of the diameter of the wire were made by men experienced in making such measurements; and the mean value was ·0371 inch, the greatest and least being ·0373 and ·0369. Using this mean value, and the directly measured torsional constant, it would seem that the modulus of rigidity of the steel is $7·71 \times 10^{11}$. As an error of ·0004 inch in the diameter measurement leads to an error of 4 per cent. in the modulus of rigidity, and as the modulus of rigidity usually taken for steel is $8·19 \times 10^{11}$, we believe that our constant 1531, as directly measured, is sufficiently correct for practical purposes.

In using (6), then, we take a to be 1531.

The virtual addition λ ought, by Maxwell's formula, to be 0·52 centim. in our case. But the bottom of the trough H was only 0·5 centim. from the edge B of the suspended cylinder in most of our experiments, and we do not know how to calculate for this. Our experiments have shown that when

this distance is 0·5 centim. the twisting moment at a given speed is practically the same as when the distance is much greater; but we did not know this from any theory, and, besides, it is always rather dangerous to depend upon a theoretical calculation of λ such as Maxwell was compelled to use. It is possible, also, that a correction of the same kind ought to be introduced for capillary and other actions at the surface of the liquid. The action of the atmosphere was in any case negligible, because when there was no liquid present in the trough, so that there was an action of the air several times greater than ever occurred during the experiments, the deflexion was quite imperceptible at much higher speeds than those used in the experiments.

The temperature being kept as nearly as possible constant, but probably varying between 18°·9 C. and 20°·1 C. (stated as 19°·5 C.), the following experiments were made with sperm-oil, beginning with a small quantity in the trough and ending with a large quantity. The bottom of the trough was in every case 0·5 centim. below the edge of the suspended cylinder.

TABLE I.—June 9th, 1892.

n.	Deflexion D when				
	l=0·5 cm.	l=2·5 cm.	l=5 cm.	l=7·5 cm.	l=9·9 cm.
50	24	67	112	158	
49	...	...	...	...	196
48					
40	18	...	...	132	163
39	...	53	91		
23	10				
17·5	8	..	...	57	73·5
17	...	24	39		
12	5	...	...	40	
11·5	...	16	27		
11	..	...	...	...	49

As the deflexion is sufficiently nearly proportional to the speed to allow of corrections by this rule when the corrections are small, we have corrected the above observations to the speeds 50, 40, 17½, 11½; and we obtain the following results :—

Values of l.	Values of n.			
	50.	40.	$17\frac{1}{2}$.	$11\frac{1}{2}$.
·5	24	18	8	5
2·5	67	54	25	16
5·0	112	93	40	27
7·5	165	132	57	$38\frac{1}{2}$
9·9	200	163	73·5	51

D and l were then plotted as the coordinates of points on squared paper ; and it was obvious that for each value of n these points lay very nearly in a straight line, and all the straight lines passed through the point $l=-0{\cdot}8$. It is curious that the linear law should hold for such small values of l as 0·5 centim., and for high speeds as well as low speeds. We shall presently see that some of these speeds are considerably above the critical speed at which (4) ceases to represent the motion.

We may take it, then, that $\lambda=0{\cdot}8$ centim., which is greater than the calculated value 0·52. The discrepance cannot be due to the distance BH being small, for we have altered this distance and found no perceptibly different results. As already stated, it may be due to some capillary surface action. Taking $a=1531$ and $\lambda=0{\cdot}8$, we have (6) becoming

$$\mu = 0{\cdot}769\,\mathrm{D}/n(l+0{\cdot}8). \quad . \quad . \quad . \qquad (7)$$

Of course our results are consistent with our equations of motion only so long as $\mathrm{D}/(l+0{\cdot}8)$ is proportional to n.

Many observations have been made with this apparatus during the last year on various liquids, under very different conditions of temperature and speed and depth. We give here a set made on sperm-oil. In all cases the bottom of the trough was 0·5 centim. below the edge of the suspended cylinder.

Keeping the oil at a constant temperature we ran the trough at a number of speeds, and repeated at other constant temperatures. The results are given in Tables IV. to XI.

When a temperature had to be taken the rotation was stopped and a thermometer dipped about halfway down in the oil, the reading being taken at the end of about half a minute. A small Bunsen flame was applied underneath the trough

when a temperature higher than the room had to be maintained for a considerable length of time.

As the temperature varied slightly, and we wished to reduce our observations to constant temperatures, we afterwards made two sets of observations at very varying temperatures but constant speeds. These later observations we shall consider first. They are given in Tables II. and III.

TABLE II.—March 29th, 1892. ($l=8{\cdot}275$ centim.)

n.	$\theta°$ C.	D.	$\frac{40}{n}\frac{D}{l+\lambda}$.	μ.	μ calculated.
39·5	13·5	189	21·09	·400	·446
39·5	17·1	165	17·60	·334	·357
40	19·5	150	16·53	·318	·317
40	24·0	134	14·77	·287	·266
40	25·3	126·5	13·94	·268	·254
40	28·9	111·5	12·28	·236	·228
40	32·0	109	11·46		
40	42·5	88	9·70		
40	46·9	81	8·65		
40	58·5	67	7·11		
40	64·0	58	6·39		
40	71·0	56	6·17		
40	77·0	50	5·51		
40	85·5	46·5	5·12		

The numbers in the column headed $\frac{40}{n}\frac{D}{l+\lambda}$ are obviously intended to be corrections of $D/(l+\lambda)$ for the constant speed of 40 revolutions per minute.

TABLE III.—March 30th, 1892. ($l=7{\cdot}78$ centim.)

n.	$\theta°$ C.	D.	$9\frac{D}{l+\lambda}$.	μ.	μ calculated.
8·75	5·2	160	19·19	1·64	2·06
9·75	8·0	85	9·15	0·78	0·82
9·2	10·0	57	6·56	0·73	0·62
9·0	10·8	47	5·48	0·47	0·56
9·0	16·6	36·5	4·26	0·363	0·366
9·2	24·8	26·5	3·02	0·258	0·259
9·0	35·0	19·5	2·27	0·194	0·196
9·0	47·0	14·0	1·63	0·139	0·137
9·0	56·5	10·2	1·19	0·101	0·104
9·0	67·0	8·0	0·93	0·079	0·081
9·0	89·0	5·4	0·63	0·060	0·058
9·0	84·5	6·0	0·70	0·054	0·054

The numbers headed $\frac{9}{n}\frac{D}{l+\lambda}$ are intended to be corrections of $D/(l+\lambda)$ for the constant speed of 9 revolutions per minute.

We have plotted the numbers in the last columns of these tables with θ upon a sheet of squared paper; but it is unnecessary to publish the resulting curves. We exhibit them to the Members of the Society.

Knowing what has been done by Prof. Osborne Reynolds, it seemed unlikely that one simple formula should satisfy either of these curves; that is, it was likely that in the lower curve there was some temperature for which the speed $n=9$ was a critical speed, and there was also a temperature for which $n=40$ was a critical speed. We therefore used the curves merely for small temperature-corrections in our other experiments, in which we kept the temperature nearly constant.

It was therefore without much interest that, in preparing this paper for publication, we tried to obtain empirical formulæ for these curves; and at first we used, not the observations themselves, but the observations as corrected by curves drawn upon squared paper.

When $\log(\theta-4\cdot2)$ and $\log\frac{D}{l+\lambda}$ are plotted as coordinates of points on squared paper, we were astonished to find that when $n=9$ the points lie in two straight lines. The allineation of the points is very striking, even when the uncorrected observations are taken, and leads to the following empirical formula:—

Letting ϕ denote $\theta-4\cdot2$, and letting y denote $D/(l+\lambda)$, the torque as measured in degrees deflexion of pointer per unit length of wetted cylinder; then, at the constant speed of $n=9$,

$$y\phi^{1\cdot349}=\text{constant for temperatures above } 40^\circ \text{ C.}$$

$$y\phi^{0\cdot686}=\text{constant for temperatures below } 40^\circ \text{ C.}$$

On plotting $\log y$ and $\log\phi$ for the constant speed $n=40$, the points are not found to lie so nicely in straight lines, but there does seem to be some sort of discontinuity at a temperature of about 45° C.

At first we thought that these temperatures were the temperatures at which the speeds $n=9$ and $n=40$ were the critical speeds, and we were greatly concerned because our result seemed to be quite out of accord with the reasoning of Prof. Osborne Reynolds *. As his ingenious theory has been completely verified by experiments made upon the very smallest and largest pipes with flowing water, and as it is simple we had adopted it for the reduction of our experiments.

According to his theory, $\frac{D}{l+\lambda}$ or, as we shall call it, y, ought to be proportional to n until n exceeds a certain value; this value being a function of μ/ρ, where ρ is the density of the fluid. Now the alteration of ρ with temperature in such a liquid as sperm-oil is so small that the error in neglecting it is small in comparison with our errors of experiment.

Neglecting, then, the alterations in ρ, the theory of Prof. Reynolds leads to

$$y = a\mathrm{F}^{2-\kappa}n^{\kappa}, \quad \ldots \ldots \quad (8)$$

where F is a function of the temperature, n the number of revolutions per minute; where $\kappa=1$ until the critical speed n_c is reached, n_c being proportional to F, and κ having a higher value than 1 for all speeds above the critical; a is a constant. This is on the assumption that Prof. Reynolds's theory would lead to the same result in our case as in his pipes.

Now, in the first place, it seemed absurd that the temperatures for which the speeds 9 and 40 were the critical speeds should be so near to one another as 40° C. and 45° C. But a much more serious consideration was this. According to any reasonable application of the theory to our case, at constant speed, if $y\phi^m$ is constant when the speed is less than the critical speed, and if $y\phi^s$ is constant when the speed is above the critical speed, then s ought to be less than m, whereas 1·349 is about twice 0·686. We came to the conclusion that the point of discontinuity has nothing whatever to do with the critical speed; indeed, we subsequently found

* "An Experimental Investigation of the Circumstances which determine whether the Motion of Water shall be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," by Osborne Reynolds, F.R.S., Phil. Trans. pt. iii. (1883).

it probable that $n=9$ does not become the critical speed until the highest temperature of Table III. is reached.

Using the deflexions in Table III. to determine μ according to (7), we have the results given in column 5 of the Tables. The numbers in column 5 of Table I. are calculated for temperatures lower than 26° C., which is about the temperature at which 40 is the critical speed. In some of the following tables, giving the results of experiments made at various constant temperatures, we have also given values of μ. There is as much consistency in all these results as might have been expected. We lay most weight upon the results given in Table III., which lead to the laws

$$\mu = 2{\cdot}06(\theta - 4{\cdot}2)^{-{\cdot}686} \text{ below } 40^\circ \text{ C.}; \quad . \quad . \quad . \quad . \quad (9)$$

$$\mu = 21{\cdot}67(\theta - 4{\cdot}2)^{-1{\cdot}349} \text{ above } 40^\circ \text{ C.} \quad . \quad . \quad . \quad . \quad (10)$$

We have searched in books in vain for a mention of a discontinuity in any other physical property of sperm-oil about this temperature; but we have already begun to experiment on its other physical properties, as it is unlikely that there should be a discontinuity in the law for the viscosity alone. At the same time, we may say that our chemical friends see no reason for a confirmation of our belief.

In the tables we give the viscosity as calculated from these formulæ; and it will be seen that they agree well enough with the observed viscosities.

TABLE IV.—March 18th, 1892.

($l=6{\cdot}15$ centim. Temperature Constant, 17°·5 C.)

n.	D.	y, or $D \div (l+\lambda)$.	μ.
36	114	16·41	·351
39	121	17·41	·344
54	172	24·75	·352
69	245	35·26	
80	300	43·16	
92	345	49·64	
23	74	10·65	·356
16	52	7·48	·360
13	41	5·90	·349
8·75	27	3·98	·349

The column headed μ is $0{\cdot}769\,D/n(l+\lambda)$, and has no

meaning at a speed greater than the critical speed. The critical speed, n_c, is probably about 50. The first three and first four values are probably measurements of μ. The average value of these seven is $\mu = 0{\cdot}351$. Formula (9) would make μ to be 0·349.

Plotting $\log y$ and $\log n$ as the coordinates of points on squared paper, the points lie very nearly in a straight line indicating $y \propto n$ until the critical speed, about $n = 50$, is reached, and for all higher speeds the points lie nearly in another straight line indicating $y \propto n^{1{\cdot}25}$.

TABLE V.—March 21st, 1892. ($l = 6{\cdot}075$ centim.)

$\theta°$ C.	n.	D.	y.	μ.
22·5	9·25	24·5	3·40	·283
23·2	11	28·0	3·97	·278
22·7	14·2	37	5·16	·280
22·0	17·2	45	6·15	·279
25·0	27	68	10·19	·277
24·5	23	57	8·41	·281
24·0	32	77	11·20	·269
23·0	90	340	48·00	
22·5	102	410	56·96	
23·0	80	276	38 94	
24·0	72	224	32·58	
24·5	56	169	24·95	
24·0	48	129	18·76	
23·5	43	108	15 42	·278
23·0	38	97	13·69	·277

The column headed y is $D/(l + \lambda)$ corrected to the constant temperature of 24° C. by a correction of about 3 per cent. per degree. The numbers in the last column have no meaning for speeds higher than $n =$ about 43. The average value of μ, the viscosity, in the first seven and last two observations is 0·278. Formula (9) would make μ for this temperature 0·266.

Plotting $\log n$ and $\log y$ on squared paper gives points lying nearly in two straight lines, showing that $y \propto n$ to the critical speed $n =$ about 43, and above that speed $y \propto n^{1{\cdot}37}$.

TABLE VI.—March 21st, 1892. ($l=6{\cdot}075$ centim.)

$\theta°$ C.	n.	D.	y.
29·5	38	81	11·63
30	43	95	13·82
31	48	114	16·99
30·5	56	134	19·73
30·5	60	147	21·65
30	66	175	25·45

y here means $D/(l+\lambda)$ corrected to the constant temperature of 30° C. by a correction of $2\frac{1}{2}$ per cent. per degree. μ as calculated from (7) would have no meaning, as the critical speed for this temperature is about 28 revolutions per minute, and we give no column headed μ. y is very nearly $\propto n^{1{\cdot}33}$.

TABLE VII.—March 22nd, 1892. ($l=5{\cdot}425$ centim.)

$\theta°$ C.	n.	D.	y.	μ.
56	38	49·5	7·92	
55	42	59	9·20	
55	54	72·5	11·31	
56	58	88	14·13	
56·5	74	116	18·79	
57·5	108	158	26·32	
58·5	29	34	5·78	
56	24	25	4·01	·1282
57	17·5	11	1·81	·0791
57	15·5	14	2·31	·1143
54	15·5	12	1·82	·0903
55	16	14	2·19	·1054

y means $D/(l+\lambda)$ corrected to 56° C. by a correction of $2\frac{1}{2}$ per cent. per degree. The critical speed is probably below $n=24$, and for speeds greater than the critical $y \propto n^{1{\cdot}32}$ nearly. For speeds less than this the average value of μ is 0·103. According to (9) the value of μ for 56° C. is 0·1055.

Plotting $\log y$ and $\log n$ as the coordinates of points on squared paper gives points which may be said to lie on two straight lines, but the errors of observation are too great. For $n>24$ we might perhaps say that $y \propto n^{1{\cdot}28}$; but it seems hardly fair to draw conclusions from this set of observations.

TABLE VIII.—March 24th, 1892. ($l=7\cdot025$ centim.)

$\theta°$ C.	n.	D.	y.	μ.
30	100	363	45·24	
30·8	78	250	31·79	
31	115	404	51·64	
31	32	62	7·92	·1903
31	22	47	6·01	·2103
31	17	37	4·73	·214
32	13	29	3·80	·2249
31	11	23·5	3·00	·2097
30	9·2	20	2·50	·209

y is $D/(l+\lambda)$ corrected to 31° C. by a correction of $2\frac{1}{2}$ per cent. per degree. The numbers of the last column have no meaning for speeds higher than n about $=32$. The average value of μ is 0·210. According to (9) the value of μ for 31° C. is 0·216.

Below $n=32$, $y \propto n$. Above $n=32$, we may perhaps say that $y \propto n^{1\cdot4}$.

TABLE IX.—March 28th, 1892. ($l=6\cdot525$ centim.)

$\theta°$ C.	n.	D.	y.
81	38·5	38	5·18
80·5	56	63	8·557
80	69	88	11·80
79·5	84	112	14·70
—	104	134	18 29
83	92	116	16·157

y means $D/(l+\lambda)$ corrected to 81° C.

The law seems to be $y \propto n^{1\cdot3}$ nearly.

TABLE X.—March 23rd, 1892. ($l=7\cdot025$ centim.)

$\theta°$ C.	n.	D.	y.
82	17·5	16	2·06
82	14	11	1·23
82	26	26	3·35
81	32	33	4·22
80	11·5	8·5	1·08
82	10	6·5	0·84

If it is assumed that $n=10$ is not much above the critical speed, μ may be calculated as 0·0646. According to (9) $\mu=0·062$ for 81° C.

$y \propto n^{1·30}$ may be taken as the law.

TABLE XI.—March 24th, 1892. ($l=6·7$ centim.)

θ° C.	n.	D.	y.	μ.
65	9	8	1·07	·091
66	10·8	9	1·21	·086
65·5	17	16	2·14	·097
64·5	21	21	2·79	
65	23·5	26	3·46	
65	29	32	4·26	
65	114	210	28·00	
66	102	162	21·82	
64·5	88	156	20·10	
65	66	112	14·93	
66	52	74	9·96	
66	44·5	59	7·94	
66	38	46	6·19	

y is $D/(l+\lambda)$ corrected to 65° C. μ has no meaning except for the first three speeds, and the mean of these three is 0·091. According to (9) the value of μ for 65° C. is 0·085.

Above the critical speed, which is possibly below $n=17$, the law is probably $y \propto n^{1·32}$.

It is not worth while to publish any of the observations which we have made upon other liquids, nor to publish the curves we have drawn for sperm-oil, although we exhibit them before the Society. Errors of one degree in observing temperature were quite possible, and errors of half a degree in the deflexion of our pointer were also possible. Small fluctuations in speed were continually taking place, so that the pointer was never quite still, the motion of the fluid was therefore not truly steady. It is our determination to repeat the whole work with improved apparatus. In the meantime, however, it will be observed from Table III. that there is fair agreement in the law connecting μ with temperature, from all the sets of observations. There is, on the whole, a very fair agreement with what we venture to call

Prof. Reynolds's rule,

$$y = a\mathrm{F}^{2-\kappa} n^{\kappa},$$

where κ has the value 1·33 or 1 according as n is above or below the critical speed*. The sheet of squared paper on which we have plotted all our values of $\log y$ and $\log n$ for the various constant temperatures shows that the errors of observation are too great for the establishment of this value of κ; but it is the probable value. It shows, however, in the allineation of the points of discontinuity, with sufficient accuracy that $y_c \propto n_c^2$, if the rule is taken to be generally true; and although there is some little vagueness always in one's observations just about the critical speed, we may take $y_c = 0{\cdot}009\, n_c^2$ without very great error. Indeed, we are satisfied with the substantial agreement of all our observations with the formula

$$y = a\left(\frac{\mu}{\cdot 769\, a}\right)^{2-\kappa} n^{\kappa},$$

where $a = {\cdot}009$ and $\kappa = 1{\cdot}33$. That is, for low speeds we have the law

$$y = \frac{\mu}{\cdot 769}\, n.$$

At the critical speed the law suddenly changes to

$$y = a\left(\frac{\mu}{\cdot 769\, a}\right)^{\frac{2}{3}} n^{\prime\frac{4}{3}},$$

which holds for all higher speeds which we have tried. The critical speed

$$n_c = \frac{\mu}{\cdot 769\, a} = 144\,\mu,$$

and

$$y_c = a n_c^2, \text{ or } {\cdot}009\, n_c^2.$$

It is to be recollected that ρ is too nearly constant for us

* Prof. Reynolds, in criticizing a proof of this paper, has been kind enough to point out that his rule for pipes does not necessarily apply to the fluid in our apparatus. We had not seen the reprint of his Royal Institution lecture, else we should have known that the condition of the liquid in circular flow is inherently stable or unstable according as r is greater or less than the radius of the fixed cylindric surface. As he points out, the liquid in the outer space is inherently stable for velocities

to say with certainty that a is proportional to ρ, as the theory requires. The errors of observation were so great that it was not worth while finding accurately the most probable values of κ and a.

We wish it to be understood that our apparatus was very carefully constructed, and great care was taken in making the observations; but it is our intention to pursue the investigation with apparatus much more carefully constructed.

Vibratory Experiments.

In designing the apparatus it was our intention to obtain μ from the damping of the rotational oscillations of the suspended cylinder about its vertical axis, the trough being at rest. We meant in this way to obtain μ for velocities very much smaller than those which could be employed in our steady motion experiments. A considerable number of observations were made, but when we tried to make calculations of μ we found that our mathematical difficulties were too great, and after many months of effort we are forced to say that we are unable to utilize these observations. In equation (4) assume that $v = w\epsilon^{ikt}$, and w may be obtained in Bessel functions. Unfortunately, as there are two surface conditions, *both* particular solutions of the Bessel equation are necessary, and the work of reduction becomes very great. An approximate solution is obtained by taking $r = \mathrm{R} + x$, R being the radius of the suspended cylinder, and taking the equation (4) to be

$$\frac{d^2v}{dx^2} + \frac{1}{\mathrm{R}}\frac{dv}{dx} - \frac{v}{\mathrm{R}^2} = \frac{\rho}{\mu}\frac{dv}{dt}. \quad . \quad . \quad . \quad (11)$$

Making this assumption in the case of steady motion, it was found that it was sufficiently correct for practical purposes. The following numbers show the sort of error introduced, taking $\mathrm{R}=10$, and 1 the greatest value of x.

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far exceeding the critical velocity (if there is one) for plane surfaces, whereas the liquid in the inner space is unstable from the first.

We directed the attention of the meeting to the fact that Tables IV., V., VI., and VIII. give unmistakable evidence of the truth of what we have called Prof. Reynolds's Rule, however difficult we may find it in explanation.

Values of x.	Values of v.		
	Correct.	Approximate.	On the assumption of motion in plane layers.
0	1	1	1
·2	·7914	·7914	·8
·5	·4875	·4872	·5
·7	·2895	·2892	·3
1·0	0	0	0

The solution of (11) for vibratory motion is easy enough; but we found it still difficult to calculate μ from our observations. Even when we assume that the motion is in plane layers, so that the solution used by Maxwell is employed, we find that our $\frac{\mu}{\rho}$ is too great for a logarithmic decrement to exist with such amplitudes and times of oscillation as we had employed in the experiments, and it was impossible for us to repeat the experiments under the same conditions again at slower velocities, because the apparatus had been taken to pieces and could not be fitted up again in exactly the same way. When we say that a logarithmic decrement did not exist, we mean that it was not constant, but varied with the amount of the oscillation. For the tractive force to be proportional to the velocity of the cylinder it is necessary for μ/ρ and the periodic time to be so great that the velocities of the fluid at all places shall be in the same proportion as if the motion were steady.

After this paper was written we asked Mr. J. B. Knight, of the Chemical Department of the Finsbury Technical College, to make measurements of the specific gravity of sperm-oil at different temperatures. His results give a very striking confirmation of the views expressed in the paper as to a discontinuity of some kind due to rise of temperature. As all the authorities whom we have consulted seemed to see no possible reason for a discontinuity in the rate of change of μ with

temperature in sperm-oil at about 40° C., it is possible that these results may be of importance.

Temperature Cent.	Specific Gravity.
25	·831
30	·8306
35	·828
40	·826
45	·8758
50	·8753
55	·8717

We are now arranging a piece of apparatus which will give, not the absolute value of the specific gravity, but with great accuracy relative rates of the change of specific gravity with temperature *. We shall make experiments of the same kind upon other animal oils.

Discussion.

After concluding the paper, Prof. Perry read a letter he had received from Prof. Osborne Reynolds on the subject, who doubted whether the true critical velocities had been reached in the experiments. In the particular arrangement employed, he would expect no critical velocity in the outer ring of liquid, whilst in the inner ring the motion would be unstable from the first.

Prof. Lodge said he had tried whether the refraction or the density of sperm-oil exhibited a discontinuity about 40°, but found none.

Mr. Rogers pointed out that experiments which corroborated those of Prof. Perry had been made by M. Couette, and published in *Ann. de Chim. et de Phys.* [6] xxi.

Mr. E. W. Smith suggested that the apparent discontinuity might be due to the separation of the constituents of an impure oil at certain temperatures and speeds of revolution.

* We described at the Meeting results obtained for other specimens of sperm-oil, with the new apparatus, which exhibited no discontinuity. Yet we can find no reason to doubt Mr. Knight's measurements.

XVI. *On the Applicability of* Lagrange's *Equations of Motion in a General Class of Problems; with especial reference to the Motion of a Perforated Solid in a Liquid.* By Charles V. Burton, *D.Sc.**

1. Let $\psi, \phi, \ldots$ be *some only* of the coordinates of a material system, so that when the values of $\psi, \phi, \ldots$ are given the whole configuration is *not* completely determinate. But suppose it known that the kinetic energy T can be expressed *as a homogeneous quadratic function of* $\dot{\psi}, \dot{\phi}, \ldots$ *only*; so that we may write

$$\left.\begin{array}{l} 2\mathrm{T} = (\psi\psi)\dot{\psi}^2 + 2(\psi\phi)\dot{\psi}\dot{\phi} + \ldots \\ (\psi\psi),\ (\psi\phi), \ldots \text{ are functions of } \psi, \phi, \ldots \text{ only} \end{array}\right\} \quad . \quad . \quad (1)$$

We also suppose it known *that* (1) *continues to hold good so long as the only (generalized) forces and impulses acting are of types corresponding to*

$$\psi, \phi, \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

2. Suppose, now, that such impulses of these types were to act on the system that $\dot{\psi}, \dot{\phi}, \ldots$ were all reduced to zero; the expression for the kinetic energy would accordingly vanish, and the system would be at rest. By supposing the last operation to be reversed, we see that *the motion at any instant could be produced from rest by impulses of the types corresponding to*

$$\psi, \phi, \ldots \textit{only}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

3. Let x, y, z be the Cartesian coordinates at time τ of a mass-element m referred to fixed axes, and let T be the kinetic energy of the system at the same instant. Further, let A be the "action" when the system moves without additional constraint from one configuration to another, and $\mathrm{A} + \delta\mathrm{A}$ the action when by workless constraints the path is slightly

* Read March 10, 1893.

modified, so that in place of the coordinates x, y, z we have $x+\delta x$, $y+\delta y$, $z+\delta z$. Then*

$$\delta \mathrm{A} = \{\Sigma m(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)\} - [\Sigma m(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)]$$
$$+ \text{ a term which necessarily vanishes}; \quad . \quad . \quad (4)$$

where [ ] and { } denote the values of the quantities enclosed at the beginning and end of the motion considered.

Suppose further that, both at the beginning and at the end, the values of $\psi, \phi, \ldots$ are the same for the one motion as for the other, so that initially and finally $\delta\psi, \delta\phi, \ldots$ are all zero. *It does not follow that all the* $\delta x, \delta y, \delta z$*'s are zero*; but

$$\Sigma m(\dot{x}\delta x + \dot{y}\delta y + \dot{z}\delta z)$$

is the so-called "virtual moment" of the actual momenta in the hypothetical displacement $\delta x, \delta y, \delta z$; that is, the virtual moment, in the same displacement, of the impulse necessary to produce the actual motion from rest. In virtue of (3), therefore, and of the initial and final vanishing of $\delta\psi, \delta\phi, \ldots$ we see that the bracketed terms of (4) must both be zero; hence

The increment $\delta\mathrm{A}$ *vanishes and* A *has a stationary value for all worklessly effected variations of path which leave the initial and final values of* $\psi, \phi, \ldots$ *unaltered.* . (5)

4. Lagrange's equations for the coordinates $\psi, \phi, \ldots$ may now be written down at once, since the investigation of Thomson and Tait† becomes applicable to the present case without modification. It will be noticed that in their equations $(10)^{\mathrm{v}}$ and $(10)^{\mathrm{vi}}$, § 327, the sign of $\partial\mathrm{V}/\partial\psi$ should be reversed.

We have thus a perfectly general proof of the proposition: *If the kinetic energy of a material system can be expressed as a homogeneous quadratic function of certain generalized velocities* $\dot{\psi}, \dot{\phi}, \ldots$ *only, the coefficients being functions of* $\psi, \phi, \ldots$ *only, and if this remains always true so long as the only forces and*

* Thomson and Tait's 'Natural Philosophy,' 2nd edit. Part I. § 327.

† *Loc. cit.*

impulses acting are of types corresponding to ψ, ϕ, . . . , the equations of motion for the coordinates ψ, ϕ, . . . may be written down from this expression for the energy, in accordance with the Lagrangian rule. Provided only that the stated conditions are satisfied, we need not consider whether the whole configuration is determined by the values of ψ, ϕ, . . . , or what is the nature of the ignored coordinates. . . . . (A)

5. Passing over the known application of this result to the motion of solids through an irrotationally and acyclically moving liquid, we come to the more general case of a perforated solid, with liquid irrotationally circulating through the apertures. Take as coordinates any six $\theta, \theta', \ldots$ which determine the position of the solid, together with $\chi, \chi', \ldots$ equal in number (m) to the apertures; each χ being the volume of liquid which, starting from a given configuration, has flowed across some one of the m geometrical surfaces, required to close the apertures, these surfaces being supposed to move along with the solid.

Of course the coordinates $\theta, \theta', \ldots \chi, \chi', \ldots$ are insufficient to determine the entire configuration of the system (including the positions of all the particles of liquid); but we shall see immediately how, in virtue of the proposition (A), Lagrange's equations may be written down.

6. Since an increment $\delta\chi$ in one of the coordinates χ is the volume of liquid which flows across a barrier-surface (*i. e.*, which flows through an aperture *relatively to the solid*), the generalized force corresponding to χ must be conceived of as a uniform pressure exerted over the said geometrical surface, by means of some immaterial mechanism attached to the solid; while the impulse corresponding to χ is of course a uniform impulsive pressure applied in the same manner. From hydrodynamical considerations we know that the measure of such an impulsive pressure is $\rho\delta\kappa$, where ρ is the density of the fluid, and $\delta\kappa$ the change produced in the circulation through the corresponding aperture.

Hence the impulses corresponding to $\chi, \chi', \ldots$ are

$$\kappa\rho, \kappa'\rho, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

where κ, κ', ... are the circulations through the various apertures.

7. Now when the motion of the liquid is irrotational, we have

$$\begin{aligned}\mathrm{T} &= \text{a homogeneous quadratic function of } \dot\theta, \dot\theta', \ldots \kappa, \kappa' \ldots \\ &\quad \text{only; coefficients functions of } \theta, \theta' \ldots \text{ only;} \\ \dot\chi, \dot\chi', \ldots &= \text{homogeneous linear functions of } \dot\theta, \dot\theta', \ldots \\ &\quad \kappa, \kappa', \ldots \text{ only; coefficients functions of } \theta, \theta', \ldots \text{ only.}\end{aligned}$$

Since the $\dot\chi$'s are equal in number to the κ's, let us suppose the last-written system of linear equations to be solved for the κ's in terms of the $\dot\chi$'s; we then have

$$\begin{aligned}\kappa, \kappa', \ldots &= \text{homogeneous linear functions of } \dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots \text{ only;} \\ &\quad \text{coefficients functions of } \theta, \theta', \ldots \text{ only.}\end{aligned}$$

Substituting in the expression for T we get

$$\begin{aligned}\mathrm{T} &= \text{a homogeneous quadratic function of } \dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots \text{ only;} \\ &\quad \text{coefficients functions of } \theta, \theta', \ldots \text{ only.}\end{aligned}$$

This, then, remains true so long as the motion of the liquid is irrotational; in other words, so long as the only forces and impulses acting are of types corresponding to $\theta, \theta', \ldots$ (since these are applied to the solid), $\chi, \chi', \ldots$ (since these are uniform over the barriers, by § 6).

If we identify $\psi, \phi, \ldots$ with the coordinates $\theta, \theta', \ldots \chi, \chi', \ldots$ of the present example, we see that the proposition (A) of § 4 is immediately applicable to this case. We may therefore ignore all other coordinates, and from the kinetic energy expressed as a function of $\dot\theta, \dot\theta', \ldots \dot\chi, \dot\chi', \ldots$ write down the Lagrangian equations for $\theta, \theta', \ldots$ and, if we wish, for $\chi, \chi', \ldots$ also. These latter, however, are less directly intelligible, since in general they involve finite pressures continuously acting over geometrical surfaces drawn through the liquid.

8. If we wish to picture the application of the principle of least action (§ 3) to the present case, we may proceed as follows:—Let the system start from the configuration (I.) and move without additional constraint or influence to the configuration (II.). Then let it start again from the configuration (1.) with the same velocities as before, and during the motion let infinitesimal additional forces act on the solid,

while infinitesimal pressures, uniform over each barrier-surface, are impressed on the liquid; the total rate at which the additional influences do work being at each instant zero. Further, let the additional influences be so adjusted that the system, after following a slightly different path, passes through a configuration such that $\theta, \theta', \ldots \chi, \chi', \ldots$ are all the same as for (II.). Then, to pass from the configuration (II.) to the present configuration requires no displacement of the solid, and only such displacement of the liquid that the *total* volume which crosses any barrier-surface is zero. In such a change of configuration impulses of the types $\theta, \theta', \ldots \chi, \chi', \ldots$ would have no "virtual moment," just as forces applied to the solid and uniform pressures applied to the barrier-surfaces would give rise to no virtual work.

9. At this stage it will be convenient to replace $\dot{\theta}, \dot{\theta}', \ldots$ by the components u, v, w of linear velocity and p, q, r of angular velocity, which determine the instantaneous motion of the solid along and about axes fixed in itself. The Lagrangian equations for the six coordinates $\theta, \theta', \ldots$ must accordingly be replaced by the forms suitable to moving axes. The expression for the energy in terms of the velocities now becomes a homogeneous quadratic function of $u, v, w, p, q, r, \dot{\chi}, \dot{\chi}', \ldots$ in which all the coefficients are known to be constants.

Let us apply the method due to Routh *, and *modify* this function with respect to the coordinates $\chi, \chi', \ldots$ If T be the value of the kinetic energy in terms of the velocities alone, the modified function (*i.e.* the kinetic part of Routh's modified Lagrangian function)

$$\mathrm{T}' = \mathrm{T} - \frac{\partial \mathrm{T}}{\partial \dot{\chi}}\dot{\chi} - \frac{\partial \mathrm{T}}{\partial \dot{\chi}'}\dot{\chi}' - \ldots = \mathrm{T} - \kappa\rho\dot{\chi} - \kappa'\rho\dot{\chi}' - \quad . \; . \; (7)$$

from (6). It is further known that the whole energy of the system

$$= \mathrm{E} + \mathrm{K}, \quad . \; . \; . \; . \; . \; . \; . \; . \; (8)$$

where E is a function of $u \ldots, p \ldots,$ only, and K is a function

* Rigid Dynamics,' vol. i. chap. viii.

of the momenta $\kappa\rho$ only. Suppose, now, that the solid were brought to rest by forces applied to it alone: E would vanish along with u, v, w, p, q, r, while the circulations κ, and consequently also K, would remain unaltered. The generalized velocities $\dot{\chi}, \dot{\chi}', \ldots$ would in general have changed, becoming, let us suppose $\dot{\chi}_0, \dot{\chi}_0', \ldots$ and the kinetic energy would accordingly have become

$$\mathrm{K}=\tfrac{1}{2}(\kappa\rho\dot{\chi}_0+\kappa'\rho\dot{\chi}_0'+\ldots). \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

Now let

$$\dot{\chi}=\dot{\chi}_0+\dot{\chi}_1, \quad \dot{\chi}'=\dot{\chi}_0'+\dot{\chi}_1', \quad . \quad . \quad . \quad . \quad . \quad (10)$$

so that each $\dot{\chi}_1$ is that part of the flux of liquid (volume per unit time) which takes place across a barrier-surface owing to the motion of the solid itself.

Having regard to (8), (9), and (10) our equation (7) for T′ becomes

$$\mathrm{T}'=(\mathrm{E}+\mathrm{K})-2\mathrm{K}-\kappa\rho\dot{\chi}_1-\kappa'\rho\dot{\chi}_1'. \quad . \quad . \quad . \quad (11)$$

Let us write for the velocity-potential of the *acyclic* motion

$$\Phi=u\phi_u+v\phi_v+w\phi_w+p\phi_p+q\phi_q+r\phi_r, \quad . \quad . \quad (12)$$

and for the value of $\dot{\chi}_1$ across the barrier-surface σ we have

$$\dot{\chi}_1=\iint\Big\{\frac{\partial\Phi}{\partial\nu}-[ul+vm+wn+p(ny-mz)$$
$$+q(lz-nx)+r(mx-ly)]\Big\}d\sigma, \quad . \quad . \quad . \quad (13)$$

where x, y, z are the coordinates of the element $d\sigma$ and l, m, n are the direction-cosines of its normal ν, all referred to the system of axes fixed in the solid. From (12) and (13) substitute in (11); thus

$$\mathrm{T}'=\mathrm{E}-\mathrm{K}+u\Sigma\kappa\rho\iint\Big(l-\frac{\partial\phi_u}{\partial\nu}\Big)d\sigma+\text{similar terms in } v, w,$$

$$+p\Sigma\kappa\rho\iint\Big(ny-mz-\frac{\partial\phi_p}{\partial\nu}\Big)d\sigma+\text{similar terms in } q, r, \quad . \quad (14)$$

where the summation refers to the m barriers.

Remembering (8) it will be seen that T' is now expressed in the proper form, namely as a function of u, v, w, p, q, r, and the momenta $\kappa\rho$, $\kappa'\rho, \ldots$ only. By means of the relations

$$\frac{d}{dt}\frac{\partial T'}{\partial u} - r\frac{\partial T'}{\partial v} + q\frac{\partial T'}{\partial w} = X, \text{ \&c., \&c.}$$

$$\frac{d}{dt}\frac{\partial T'}{\partial p} - w\frac{\partial T'}{\partial v} + v\frac{\partial T'}{\partial w} - r\frac{\partial T'}{\partial q} + q\frac{\partial T'}{\partial r} = L, \text{ \&c., \&c.}$$

the equations of motion of the solid can at once be written down. X, . . ., L, . . ., are of course impressed force-and couple-constituents.

10. Since the kinetic energy due to any number of perforated solids, moving in circulating liquid, can be divided into two parts, of which one is a function of the component velocities of the solids alone, and the other a function of the circulation-momenta alone, the above method may obviously be extended ; in fact a slight change in (14) will render it at once applicable to the more general case. We shall have, evidently,

$$T' = E - K + \Sigma u \Sigma \kappa\rho \iint \left(l - \frac{\partial \phi_u}{\partial \nu}\right) d\sigma + \text{similar terms in } v, w,$$

$$+ \Sigma p \Sigma \kappa\rho \iint \left(ny - mz - \frac{\partial \phi_p}{\partial \nu}\right) d\sigma + \text{similar terms in } q, r, \quad (15)$$

where E is still the energy due to the motion of the solids and the acyclic motion of the liquid, and K the energy due to the circulations. In each barrier-term the first Σ denotes summation with respect to all the solids, and for each u or p, &c., the second Σ denotes summation with respect to all the barriers of the system.

These hydrodynamical results are not new, but the method of proof is in some respects different from anything that has yet been given, and will, I hope, be found intelligible and fairly simple. In an admirable memoir, just communicated to the Physical Society, Mr. Bryan has given a direct hydrodynamical proof of the equations holding good for the motion of the system in question ; but it seemed to me also desirable that the problem should be rigorously treated by the method of

generalized coordinates, avoiding any assumption as to the impulse of the cyclic motion, and proceeding entirely from the principles established by Lagrange, and extended by Hamilton, Routh, and Hayward.

When this paper was in proof it contained some remarks on the ignoration of coordinates, as treated in Thomson and Tait's 'Natural Philosophy'*. Calling $\chi, \chi', \ldots$ the independent coordinates which, together with $\psi, \phi, \ldots$ determine the whole configuration of the system in §§ 1, ..., it was suggested that, in hydrodynamical and kindred applications, there was a difficulty in proving that $\partial T/\partial \chi$, $\partial T/\partial \chi', \ldots$ were all zero.

But the difficulty, if indeed it should exist, is easily removed. For since the actual motion at any instant could be generated from rest by impulses of types corresponding to $\psi, \phi, \ldots$ only, we have throughout the motion

$$\partial T/\partial \dot{\chi}=0, \quad \partial T/\partial \dot{\chi}'=0, \ldots;$$

and by the Lagrangian equations for $\chi, \chi', \ldots$, since all the generalized forces are of types corresponding to $\psi, \phi, \ldots$, we get

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\chi}}-\frac{\partial T}{\partial \chi}=0, \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{\chi}'}-\frac{\partial T}{\partial \chi'}=0, \ldots$$

whence

$$\frac{\partial T}{\partial \chi}=0, \quad \frac{\partial T}{\partial \chi'}=0, \ \ldots$$

Discussion.

A criticism by Mr. A. B. Basset on Mr. Bryan's recent paper, and also on Dr. Burton's paper, was read by Mr. Elder.

Mr. Basset regards the process employed by Mr. Bryan in obtaining the equations of motion as a distinctly retrograde step, and thinks the most scientific way of dealing with dynamical problems is to avoid the unnecessary introduction of

* Part I. § 319, example G.

any unknown reactions. The advantages of the theory of the impulse are described by Mr. Basset, and the parts which require care when applying the theory to cyclic irrotational motion pointed out. Comparisons are then made as regards simplicity, between the different methods of treating the subject which have been used by Mr. Bryan, Prof. Lamb, and himself.

With reference to Dr. Burton's paper, he thinks it will tend to complicate rather than elucidate the subject.

An account of how Lagrange's original equations had been modified by Hamilton, Routh, and himself is given at some length, and the advantages and power of the mixed transformation which he had developed are pointed out.

Prof. Henrici said he agreed with Mr. Basset, in preferring the more general method, but thought the independent treatment of special problems, as given by Mr. Bryan and Dr. Burton, very desirable.

Dr. Burton, in reply, said he concurred with Mr. Basset on some points, but thought it decidedly advantageous to look at problems from different points of view. The investigation he (Dr. Burton) had given was applicable to any number of solids, and, on the whole, simpler than Mr. Basset's.

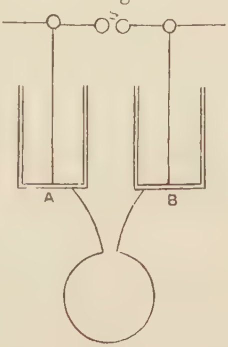
The President pointed out that no attack had been made on the validity or accuracy of Mr. Bryan's or Dr. Burton's work. As to simplicity of the various methods, different opinions might be expected to exist. He himself thought it very desirable that such problems should be approached from different sides.

XVII. *Luminous Discharges in Electrodeless Vacuum-Tubes.* *By* E. C. Rimington*.

Since reading a paper in conjunction with Mr. E. W. Smith on November 25th, 1892, before this Society†, on "Experiments in Electric and Magnetic Fields, Constant and Varying," the Author's attention has been drawn to a paper contributed by Mr. Tesla to the 'Electrical Engineer' of New York, July 1st, 1891, in which the luminous ring-shaped discharge obtained when a Leyden jar is discharged through a coil of wire surrounding an exhausted bulb is attributed to the electrostatic action of the surrounding wire, and not to the electric stress set up in the rarefied dielectric in consequence of the rapidly oscillating magnetic induction through the bulb.

As one experimental proof of this assertion Mr. Tesla gives the following experiment :—"An ordinary lamp-bulb was surrounded by one or two turns of thick copper wire, and a luminous circle excited by discharging the jar through this primary. The lamp-bulb was provided with a tinfoil coating on the side opposite to the primary, and each time the tinfoil coating was connected to the ground, or to a large object, the luminosity of the circle was considerably increased."

Fig. 1.



The author repeated this experiment with two Leyden jars arranged as in fig. 1, and found that when the spark-gap was sufficiently large to produce a bright ring when the tinfoil was not connected to earth, doing so produced no noticeable difference in the brilliancy ; but that, if the discharge were faint, it was renrendered considerably brighter on making the earth connexion. Better results were, however, obtained on connecting the tinfoil to either of the outside coatings, A or B, of the jars instead of

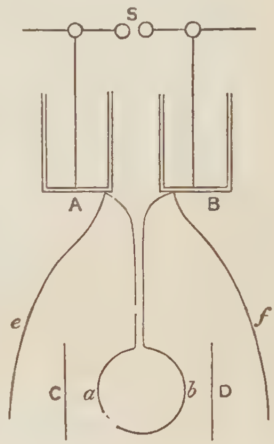
* Read April 28, 1893.

† *Ante*, p. 3.

to earth. This result led the author to try a series of experiments to endeavour to determine the cause of the effect, of which the typical ones are here given.

Experiment 1 (*vide* fig. 2). A and B are the outside coatings of a pair of Leyden jars (those employed were about pint size). C and D two vertical and parallel metal plates, at a distance of about one foot from the jars. The spark-gap, S, is adjusted by a screw, so that the spark-length can be varied by small amounts when necessary. A single turn of wire, *a b*, encloses an exhausted bulb, and its ends are connected to A and B, as shown in the figure, so that *a* the part nearest to C is connected to A, and *b* to B. Two loose wires, *e* and *f*, are also connected to A and B.

Fig. 2.



The spark-gap is now shortened until there is just no luminous ring in the bulb.

The plates C and D are then connected to the outer coatings A and B by means of the two loose wires, with the following results :—

(1) A to C. Bright ring.
(2) B to D. Bright ring.
(3) A to C and B to D simultaneously. Bright ring.
(4) A to D.
(5) B to C.
(6) A to D and B to C.
} No luminous ring.

Expt. 2.—The wire turn *a b* is removed from the bulb, given a half twist, and then replaced ; so that *a* is now nearest to D, and *b* to C. Plates not connected, no luminous ring.

(1) A to C.
(2) B to D.
(3) A to C and B to D.
} No luminous ring.

(4) A to D.
(5) B to C. } Bright ring.
(6) A to D and B to C.

Expt. 3.—Arranged as in Expt. 1, case (1) or (2). C is then connected to D, and the ring becomes less bright.

Expt. 4.—Arranged as in Expt. 1, case (1). C and D connected. On approaching C to the bulb, ring becomes brighter.

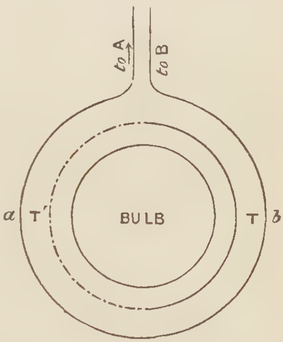
On approaching D less bright.

If arranged as Expt. 1, case (2), the reverse happens.

All the above four experiments give the same effects if the turn of wire be larger than the bulb, as in fig. 3, only a longer spark-gap has to be used.

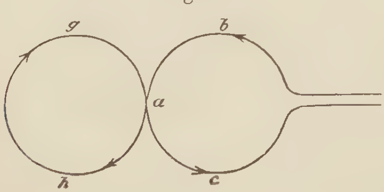
Expt. 5.—A single turn of wire (fig. 3), *a b*, larger than the bulb is employed, and between the bulb and the ring a semicircular strip of tinfoil or metal T is placed. The wire is connected as in Expt. 1. The spark-gap is arranged to give no ring. Connecting T to B bright ring, T to A no ring. The reverse happens if the tinfoil is placed in position T′ as shown by the dotted line.

Fig. 3.



Expt. 6.—A piece of gutta-percha covered wire is bent into shapes shown in figs. 4 and 5. On placing either of these over bulb as in fig. 6, a figure-of-eight-shaped luminous

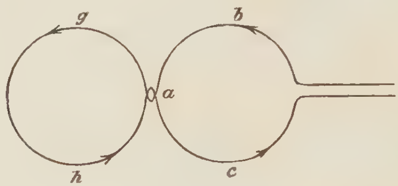
Fig. 4.



discharge is obtained, and there is no noticeable difference between the two.

Expt. 7.—Putting the wire (fig. 4) on bulb as in fig. 7, a single broad band-ring is obtained, as the two turns will help one another with respect to magnetizing effect.

Fig. 5.



Doing the same with the wire (fig. 5) a discharge is obtained shaped like the sector of an orange, as shown by the dotted lines, fig. 7.

Fig. 6.

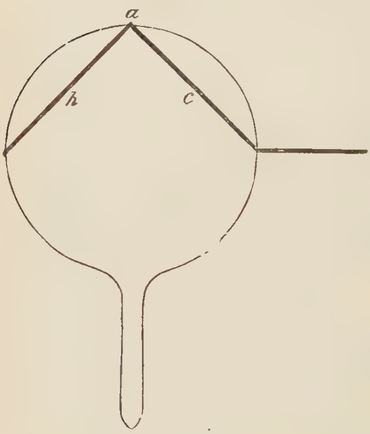
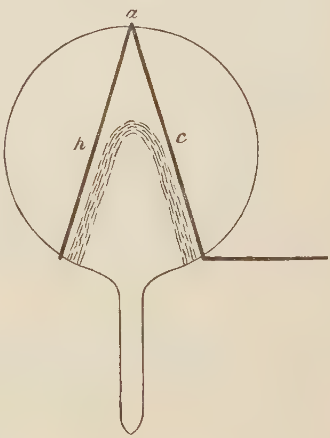


Fig. 7.



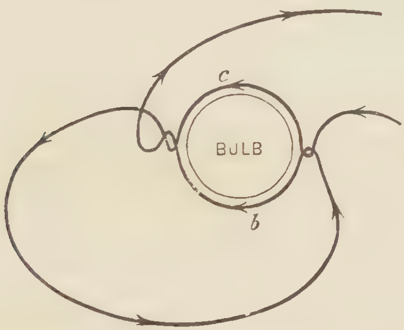
Expt. 8.—Bending a wire as shown in fig. 8, and placing a bulb in the loop *b c*, there is no effect even with a long spark-gap, although the potential difference between the sides *c* and *b* would be much greater than in the case of a single turn.

Putting a bulb in the loop, *b c*, of fig. 4 at once gives a bright ring.

Experiments 6, 7, and 8 seem to show that ring, or other shaped, sharp luminous discharges can only be obtained with

the wire so wound as to give magnetic induction through the bulb, while the first five experiments show that an electrostatic field in the bulb may help the effect. The theory the

Fig. 8.



author has come to after consideration of the above and other experiments is :—That if the E.M.F. due to rate of change of magnetic induction acting in the dielectric of rarefied gas be insufficient to break it down and produce a luminous discharge (owing to the spark-gap being too short), the electrostatic field between the plates C and D, or between one of the plates and part of the wire, if correctly timed with respect to the rate of change of current in the wire, will commence the breakdown of the gas, thus allowing a less E.M.F. due to the magnetic induction to complete it.

To put this to the test, a single turn of wire was put round a bulb and the spark-gap adjusted so as to give a very faint or no luminous ring ; on the top of the bulb was laid a piece of tinfoil connected to one pole of a $\frac{1}{4}$ in. spark induction-coil ; when the coil is worked the tube is filled with a faint glow : if now the Leyden jars are charged and discharged there will be sometimes a ring in the bulb which will be occasionally quite bright. The reason it cannot be always bright is of course that the discharges of the induction-coil are periodic, as are also those of the jars, and it is only when the two are properly timed (*i.e.* the P.D. due to the coil coming either just before or simultaneously with the spark) that there will be a bright ring.

This experiment seems to settle the question and show conclusively that a properly timed electric stress in the bulb due to an electrostatic field will allow an E.M.F. due to the alternating current in the wire to produce a breakdown of the rarefied gas, which the latter is too small to effect without the aid of the former*.

In Expt. 1, when A and C are connected this field will exist between C and b the side of the turn of wire remote from C, and must therefore pass through the bulb. When A is connected to D, as the strongest field is between b and D, where the P.D. is greatest, it does not pass through the bulb; in fact the field in the bulb will simply be that due to the P.D. between a and b, or the same as it is if the wires e and f are disconnected. The results of Experiments 2, 3, 4, and 5 are also obviously explained by this theory.

To treat the subject mathematically. We have the well-known equations for the discharge of a condenser:

$$\mathrm{L}\frac{dc}{dt} + \mathrm{R}c = -\frac{q}{\mathrm{K}}, \text{ where K is the capacity,}$$

and

$$c = \frac{dq}{dt}.$$

Combining these,

$$\frac{d^2q}{dt^2} + \frac{\mathrm{R}dq}{\mathrm{L}dt} + \frac{1}{\mathrm{KL}}q = 0.$$

To obtain an oscillatory discharge 4L must be greater than KR^2.

* To prevent misconception, it had better be definitely stated that this electrostatic stress does not necessarily act in the same direction as the E.M.F. due to the rate of change of magnetic induction. In experiments (1) to (5) the direction of the former will be through the bulb from side to side, while that of the latter is a circle coplanar with the wire. As the discharge in a gas is of a nature more or less electrolytic, being accompanied by the splitting up of the molecules, it seems reasonable to suppose that anything which increases the number of dissociated molecules will enable a smaller stress to produce a breakdown in the form of a luminous discharge.

Putting a for $-\frac{R}{2L}$ and b for $\sqrt{\frac{1}{KL}-\frac{R^2}{4L^2}}$, the solution is

$$q=Qe^{at}\ \frac{\sqrt{a^2+b^2}}{b}\sin(bt+\theta), \quad . \quad . \quad . \quad (1)$$

where $\theta=\tan^{-1}\left(-\frac{b}{a}\right)$ and Q is the initial charge.

This may be more conveniently written

$$q=Qe^{at}\frac{\sqrt{a^2+b^2}}{b}\cos(bt-\eta),$$

where

$$\eta=\frac{\pi}{2}-\theta, \text{ or } \tan\eta=-\frac{a}{b}=\sqrt{\frac{KR^2}{4L-KR^2}}.$$

If the oscillations are to be rapid, $\frac{1}{KL}$ must be large compared to $\frac{R^2}{4L^2}$.

Therefore η will be some small angle.

Instead of quantity we may write P.D. of the condenser, or

$$v=Ve^{at}\ \frac{\sqrt{a^2+b^2}}{b}\cos(bt-\eta). \quad . \quad . \quad . \quad (2)$$

The current

$$c=\frac{dq}{dt}=\frac{Q}{bKL}e^{at}\sin bt=\frac{V}{bL}e^{at}\sin bt. \quad . \quad . \quad . \quad (3)$$

Now the electric stress acting in the bulb is proportional to the rate of change of current, or to $\frac{dc}{dt}$;

and
$$\frac{dc}{dt}=\frac{V}{bL}e^{at}(a\sin bt+b\cos bt). \quad . \quad . \quad . \quad . \quad (4)$$

The current itself will be a maximum or minimum when

$$\frac{dc}{dt}=0;$$

i. e. when
$$a\sin bt+b\cos bt=0,$$

or when
$$\tan bt=-\frac{b}{a}=\sqrt{\frac{4L}{KR^2}-1}.$$

Therefore $bt = \theta$, and is in general nearly equal to $\frac{\pi}{2}$.

The maximum values of the current occur when

$$bt = \theta, \quad 2\pi + \theta, \quad 4\pi + \theta, \text{ \&c.},$$

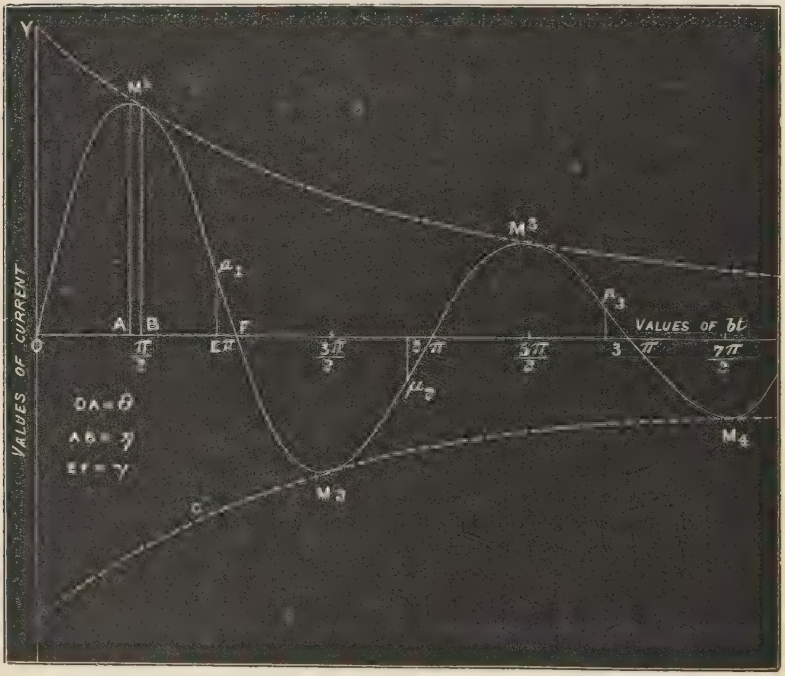
and the minimum values when

$$bt = \pi + \theta, \quad 3\pi + \theta, \quad 5\pi + \theta, \text{ \&c.}$$

This is shown in the curve (fig. 9), the points M_1, M_2, M_3, &c., representing the maximum and minimum values of the current. The distance O A represents θ, and $AB = \frac{\pi}{2} - \theta = \eta$.

Fig. 9.

Dotted curves are values of the exponential $\frac{V}{bL} e^{-\frac{R}{2L}t}$.



M_1, M_2, M_3, M_4 are the maximum values of the current. μ_1, μ_2, μ_3 are the points where the rate of change of current is greatest.

It is now necessary to consider when the rate of change of

the current is greatest. $\frac{dc}{dt}$ will be a maximum or minimum when $\frac{d^2c}{dt^2}=0$.

Now

$$\frac{d^2c}{dt^2}=\frac{V}{bL}e^{at}\{(a^2-b^2)\sin bt+2ab\cos bt\}=0.$$

Hence

$$\tan bt=-\frac{2ab}{a^2-b^2}=-\frac{R\sqrt{K(4L-KR^2)}}{2L-KR^2}.$$

Let

$$\tan^{-1}\frac{R\sqrt{K(4L-KR^2)}}{2L-KR^2}=\gamma.$$

γ will be in general a small angle.

The rate of change of current will be greatest (either a maximum or a minimum) when

$$bt=-\gamma,\ \ \pi-\gamma,\ \ 2\pi-\gamma,\ \&c.$$

Obviously bt cannot equal $-\gamma$, so that the rate of change of current is greatest for values $\pi-\gamma$, $2\pi-\gamma$, &c.; or at points μ_1, μ_2, μ_3, &c. in the curve (fig. 9), and $EF=\gamma$. If the oscillations are to be very rapid KR^2 must be negligible compared to $4L$; in which case

$$\tan\gamma=R\sqrt{\frac{K}{L}};$$

also

$$\tan\eta=\frac{R}{2}\sqrt{\frac{K}{L}},$$

and they are both very small angles, hence $\gamma=2\eta$ approximately, or $EF=2AB$.

When $bt=\pi-\gamma$,

$$\frac{dc}{dt}=\frac{V}{L}e^{-\frac{\pi-\gamma}{\sqrt{\frac{4L}{KR^2}-1}}},$$

or if the oscillations are very rapid,

$$\frac{dc}{dt}=\frac{V}{L}e^{-\frac{\pi R}{2}\sqrt{\frac{K}{L}}}.$$

If, however, $t=0$,

$$\frac{dc}{dt}=\frac{V}{L},$$

so that the greatest rate of change of current occurs at the first instant of discharge, although this is not a mathematical maximum.

Equation (4) may also be written

$$\frac{dc}{dt}=\frac{V\sqrt{a^2+b^2}}{bL}e^{at}\cos(bt+\eta), \quad . \quad . \quad . \quad (5)$$

η being the same angle as before.

It is now necessary to consider the values of the P.D. between the outside coatings A and B of the Leyden jars.

Let l and r be the inductance and resistance of the coil connected to the outer coatings, and L and R the same for the whole circuit. Let $v_1-v_2=x$ be the P.D. between the outer coatings at any instant t. Then

$$c=\frac{V}{bL}e^{at}\sin bt,$$

and

$$x=v_1-v_2=cr+l\frac{dc}{dt},$$

$$\therefore \quad x=\frac{V}{bL}e^{at}\{(r+la)\sin bt+lb\cos bt\}$$

$$=\frac{V}{bL}e^{at}\sqrt{l^2b^2+(r+la)^2}\cos(bt-\eta') \quad . \quad . \quad (6)$$

where

$$\tan\eta'=\frac{\frac{r}{l}+a}{b}.$$

η' will be in general a small angle not very different from η; and if $\frac{l}{r}=\frac{L}{R}$ or the time-constant of the coil equals the time-constant of the whole circuit,

$$\frac{r}{l}=\frac{R}{L}=-2a,$$

$$\therefore \tan\eta'=-\frac{a}{b}=\tan\eta, \text{ or } \eta'=\eta.$$

That is x is in phase with v the P.D. at the inner coatings of the jars.

To find the maxima and minima of x we have

$$\frac{dx}{dt}=\frac{\mathrm{V}}{b\mathrm{L}}e^{at}\left[\{ar+l(a^2-b^2)\}\sin bt+b(r+2la)\cos bt\right]=0.$$

$$\therefore\ \tan bt=-\frac{b(r+2la)}{ar+l(a^2-b^2)}$$

$$=\frac{(r\mathrm{L}-\mathrm{R}\quad\sqrt{\mathrm{K}(4\mathrm{L}-\mathrm{KR}^2)}}{\mathrm{L}(\mathrm{KR}r+2\mathrm{L})}$$

$$=\tan\delta.$$

Then x has its greatest positive or negative values when $bt=\delta,\ \pi+\delta,\ 2\pi+\delta$, &c. δ is in general a small angle, and is positive if

$$\frac{\mathrm{L}}{\mathrm{R}}>\frac{l}{r},$$

and negative if

$$\frac{\mathrm{L}}{\mathrm{R}}<\frac{l}{r}.$$

If

$$\frac{\mathrm{L}}{\mathrm{R}}=\frac{l}{r};\quad\delta=0.$$

If $\frac{\mathrm{L}}{\mathrm{R}}$ be not greater than $\frac{l}{r}$ the first largest value of x will occur at $t=0$, and as the rate of change of current is also greatest at this instant the two will occur simultaneously.

The next greatest value of x occurs when

$$bt=\pi-\delta\left(\text{if}\,\frac{\mathrm{L}}{\mathrm{R}}<\frac{l}{r}\right),$$

and the next greatest rate of change of current when

$$bt=\pi-\gamma.$$

δ will be less than γ, if $\frac{\mathrm{L}}{\mathrm{R}}$ be nearly equal to $\frac{l}{r}$; so that the maximum value of x will occur after the maximum of $\frac{dc}{dt}$, but the value of x will not differ very much from its maximum

when $\frac{dc}{dt}$ is a maximum *. This bears out the results obtained in experiments 1 and 2, though, of course, the electric field in the bulb will be that due to the P.D. between one of the plates, C or D, and the opposite side of the turn of wire, and this will only be about half that between the outer coatings A and B. Moreover, the phase of the potential of C will not be quite the same as that of A, on account of the inductance of the connecting wire *e*. Experiments 1 and 2 were, however, tried with the plates C and D, and the connecting wires removed, the turn of wire *ab* being moved so as to bring either *a* or *b* nearest to A or to B, and the results obtained were practically the same as those of experiments 1 and 2.

Effect of Size of Jars.

When different-sized Leyden jars are employed with the same length of spark-gap the luminous ring is more brilliant

* The above investigation into the value of the P.D. between the outer coatings will only give correctly the state of things when a steady swing has been set up in the circuit; as evidently when $t=0$ the value of x also equals zero, so that x must start in phase with the current; it will, however, rapidly get out of phase with the latter, and finally be nearly in quadrature with it. This is due to an initial wave starting from the spark-gap which runs round the circuit. Possibly the value of can be empirically represented by one of the two subjoined formulæ :—

$$x=\frac{\mathrm{V}e^{at}}{b\mathrm{L}}\sqrt{l^2b^2+(r+la)^2}\sin\{(bt+\psi)(1-e^{-pt})\},$$

or

$$x=\frac{\mathrm{V}e^{at}}{b\mathrm{L}}\sqrt{l^2b^2+(r+la)^2}\sin\{bt+\psi(1-e^{-pt})\},$$

where $\psi=\frac{\pi}{2}-\eta'$, and p some constant. Dr. Lodge, in his researches on the A and B sparks, approximately represents the initial values of x by the current multiplied by the impedance of the conductor r, or makes

$$x=\frac{\mathrm{V}e^{at}}{b\mathrm{L}}\sqrt{l^2b^2+r^2}\sin bt.$$

The initial maximum of x will consequently roughly coincide with the maximum of the current, or be near the point M_1 of fig. 9, and will thus come about a quarter of a period before the second maximum rate of change of current, point μ_1 (fig. 9).

with larger jars. Now the E.M.F, acting in the rarefied gas, and producing the breakdown of the same, is proportional to $\frac{dc}{dt}$.

Also the greatest value of $\frac{dc}{dt}$ that first occurs is when $t=0$, and then

$$\frac{dc}{dt}=\frac{\mathrm{V}}{\mathrm{L}},$$

and the next is for very rapid oscillations

$$\frac{dc}{dt}=\frac{\mathrm{V}}{\mathrm{L}}e^{-\frac{\pi \mathrm{R}}{2}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}.$$

So that the first value of the E.M.F. acting in the gas is independent of the capacity, and the next and succeeding values are less the greater the capacity.

The effect on the eye, however, of the luminous ring will be the time-integral of the discharge or approximately depend on

$$\int_0^\infty \frac{dc}{dt}\,dt.$$

The whole limits of t, viz., from 0 to ∞, cannot be taken at once, as $\frac{dc}{dt}$ keeps reversing, and this reversal will not affect the luminous discharge. Referring to the curve (fig. 9) it will be seen that the first reversal must take place at M_1, when $bt=\theta$, and subsequent ones for values $\pi+\theta$, $2\pi+\theta$, &c., of bt. It is therefore necessary to take first the limits θ and 0, then $\pi+\theta$ and θ; $2\pi+\theta$ and $\pi+\theta$, and so on, alternately writing the integrals plus and minus.

$$\int_0^\infty \frac{dc}{dt}\cdot dt=\frac{\mathrm{V}}{b\mathrm{L}}\left\{\,{}_0^{\theta}\left[e^{at}\sin bt\right]-{}_{\theta}^{\pi+\theta}\left[e^{at}\sin bt\right]\right.$$
$$\left.+{}_{\pi+\theta}^{2\pi+\theta}\left[e^{at}\sin bt\right]-\text{\&c. ad inf.}\right\}.$$

Remembering that

$$\begin{aligned}\sin(\pi+\theta)&=-\sin\theta,\\ \sin(2\pi+\theta)&=\sin\theta,\\ \sin(3\pi+\theta)&=-\sin\theta,\ \text{and so on,}\end{aligned}$$

this gives

$$\int_0^\infty \frac{dc}{dt} \cdot dt = \frac{2\mathrm{V}}{b\mathrm{L}} \sin\theta \left\{ e^{\frac{a}{b}\theta} + e^{\frac{a}{b}(\pi+\theta)} + e^{\frac{a}{b}(2\pi+\theta)} + \&\text{c.} \right\}.$$

The series in the bracket is a geometrical progression, in which the constant factor is $e^{\frac{a}{b}\pi}$; and, since a is negative, this is less than unity.

Hence

$$\int_0^\infty \frac{dc}{dt}\, dt = \frac{2\mathrm{V}\sin\theta}{b\mathrm{L}} \cdot \frac{e^{\frac{a}{b}\theta}}{1-e^{\frac{a}{b}\pi}},$$

and

$$\tan\theta = \sqrt{\frac{4\mathrm{L}-\mathrm{KR}^2}{\mathrm{KR}^2}}, \text{ or } \sin\theta = \sqrt{\frac{4\mathrm{L}-\mathrm{KR}^2}{4\mathrm{L}}};$$

also

$$b = \sqrt{\frac{4\mathrm{L}-\mathrm{KR}^2}{4\mathrm{KL}^2}}, \text{ and } \frac{\sin\theta}{b} = \sqrt{\mathrm{KL}};$$

$$\therefore \int_0^\infty \frac{dc}{dt}\, dt = 2\mathrm{V}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}} \cdot \frac{e^{\frac{a}{b}\theta}}{1-e^{\frac{a}{b}\pi}}$$

$$= \frac{2\mathrm{V}\sqrt{\frac{\mathrm{K}}{\mathrm{L}}}}{e^{\theta\sqrt{\frac{\mathrm{KR}^2}{4\mathrm{L}-\mathrm{KR}^2}}} - e^{-(\pi-\theta)\sqrt{\frac{\mathrm{KR}^2}{4\mathrm{L}-\mathrm{KR}^2}}}}.$$

Let $\theta\sqrt{\frac{\mathrm{KR}^2}{4\mathrm{L}-\mathrm{KR}^2}} = x$ and $\pi\sqrt{\frac{\mathrm{KR}^2}{4\mathrm{L}-\mathrm{KR}^2}} = y$.

Then the denominator $= e^x - e^{x-y}$, and

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \&\text{c.},$$

$$e^{x-y} = 1 + x - y + \frac{x^2}{\lfloor 2} - \frac{2xy}{\lfloor 2} + \frac{y^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} - \frac{3x^2y}{\lfloor 3} + \frac{3xy^2}{\lfloor 3} - \frac{y^3}{\lfloor 3} + \&\text{c.},$$

$$e^x - e^{x-y} = y + \frac{2xy}{\lfloor 2} - \frac{y^2}{\lfloor 2} + \frac{3x^2y}{\lfloor 3} - \frac{3xy^2}{\lfloor 3} + \frac{y^3}{\lfloor 3}$$

$$+ \text{ terms of the 4th, 5th, \&c. powers.}$$

Now x and y will in general be small fractions, since KR^2 is usually much less than $4L$.

If the oscillations are very rapid, θ is very nearly equal to $\frac{\pi}{2}$. Hence $y=2x$ approximately. Then e^x-e^{x-y} becomes

$$2x+\frac{2x^3}{6}=2x\left(1+\frac{x^2}{6}\right) \text{ approx.}$$

Therefore the time-integral

$$=\frac{2V\sqrt{\frac{K}{L}}}{2x\left(1+\frac{x^2}{6}\right)}=\frac{V}{x}\sqrt{\frac{K}{L}}\left(1-\frac{x^2}{6}\right) \text{ approx.,}$$

and

$$x=\frac{\pi}{2}\sqrt{\frac{KR^2}{4L-KR^2}}=\frac{\pi R}{4}\sqrt{\frac{K}{L}} \text{ approx.;}$$

so time-integral

$$=\frac{4V}{\pi R}\left(1-\frac{\pi^2R^2K}{96L}\right) \text{ approx.}$$

Now from this it is seen that the effect of increasing the capacity would be to slightly diminish the time-integral, and consequently probably make the brilliancy of the luminous discharge less, if it were not that increasing the capacity diminishes the real resistance of the circuit, since it makes the oscillations slower, and the resistance R for copper for rapid oscillations approximately equals $\sqrt{\frac{1}{2}blR_0}$; where l is the length of the wire, and R_0 its resistance for steady currents. Now $b=\frac{1}{\sqrt{KL}}$ approximately.

Therefore

$$R=\sqrt{\frac{lR_0}{2\sqrt{KL}}};$$

so that the time-integral is very roughly proportional to the fourth root of capacity.

There is also another reason why larger jars might produce a brighter discharge, even though the time-integral were less.

With larger jars the time taken for the amplitude of the current to sink to a value at which it becomes insignificant will be longer than in the case of small ones. Now, as the initial value of $\frac{dc}{dt}$ is the same whatever the size of the jars, the after values (although their time-integral is less and their actual values less also) last longer in the case of larger jars.

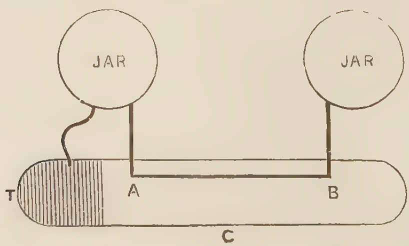
When the breakdown of the dielectric of rarefied gas is once begun by the initial $\frac{dc}{dt}$, the values of $\frac{dc}{dt}$ necessary to keep it up may probably be very much less, and consequently the smaller values of $\frac{dc}{dt}$ lasting longer, as given by the larger jars, may produce a luminous discharge more brilliant to the eye than the larger values of $\frac{dc}{dt}$ lasting a shorter time, as given by the smaller jars.

The actual results obtained with a ring of four turns of wire containing an exhausted bulb about $2\frac{1}{2}$ inches in diameter were that the differences in brilliancy, obtained by using half-gallon jars, pint jars, or very small jars made from specimen glasses, were not so very great.

Other Effects. Apparently unclosed Discharges.

A closed luminous discharge is not the only one that can be obtained. Mr. Tesla, in 1891, pointed out that by wrapping a wire round an exhausted tube so as to form a coarse-

Fig. 10.



pitched spiral, a luminous spiral discharge is obtained. He was apparently only able to obtain a very feebly luminous

spiral, but the author has succeeded in getting one quite as brilliant as in the case of the ring-shaped discharge obtained with a bulb.

In fig. 10 two half-gallon jars have their outer coatings connected by a wire, A B, bent as shown in the figure. Over the wire is laid an exhausted tube, C, with a tinfoil cap*, T, at one end; T is connected to the outer coating of the jar nearest to it. The object of this is to utilize the electrostatic effect and make the tube more sensitive to breakdown by the electromagnetic one. When the jars discharge, a straight luminous band is observed in the tube directly over A B.

If the tube C be now moved towards the jars, even by a very small amount, a closed circuit discharge will be obtained. There is apparently, then, a tendency for the luminous discharge to form a closed circuit whenever possible; and it seems probable that even when the discharge is apparently not closed, as in the case of the spiral or the straight line, the electric stress acting in the rarefied gas takes the form of a closed circuit, but is only intense enough to produce sharp luminosity close to the wire†. To further test the question an unclosed ring tube was made, and when it was placed inside a coil of wire no trace of a single luminous band could be seen‡. A small glass tube was also bent so as to form a spiral of four turns, and exhausted. A wire following the spiral was attached to it, but this also gave no trace of luminous discharge.

* It is not always necessary to use this cap, as, if the exhaustion is high enough to give green phosphorescence of the glass, with the two half-gallon jars in series, the discharge can be obtained without the cap. With another tube of lower vacuum the author finds the cap necessary.

† That is, the return part of the discharge is so diffused and feebly luminous as to easily pass unnoticed in comparison with the sharp and brilliant luminosity close over the conductor. The same applies to the spiral discharge, each turn of the spiral probably forming a closed circuit by itself.

‡ On afterwards repeating this experiment the author obtained a discharge in parts of the tube, and with half-gallon jars in the whole tube. The discharge, however, was a closed one, as there were two distinct bands in the tube, one on the side next to the coil and the other on the side farthest away from it. This is what might be expected if the magnetic induction be sufficiently strong.

Magnetic Effects of Discharge.

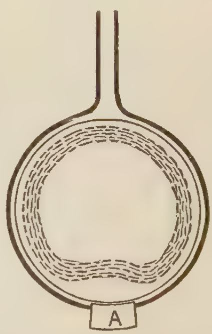
The ring discharge in a bulb or closed circular tube acts like a metallic circuit as far as magnetic effects are concerned. This may easily be shown by the following experiment.

A coil of three or four turns of wire has a similar one wound with it to form a secondary; the latter is connected to a third coil, in which is placed an exhausted bulb. The first coil is connected to the outside coatings of the jars (fig. 1). The spark-gap can be adjusted so that a fairly bright ring is produced in the bulb. If now a second bulb is placed within the first coil a luminous ring will be formed in it, and the ring in the other bulb will be much weakened or altogether extinguished. Exactly the same effect is produced if a metal plate or closed coil be brought near the first coil in lieu of the bulb.

Sensitive State of Discharge.

If a single turn of insulated wire surround one of the exhausted bulbs as in fig. 1, and the spark-gap be adjusted so as to produce a rather faint luminous ring (the fainter the better); on approaching the finger and touching the wire at any point the discharge appears to be repelled, and takes the shape shown in fig. 11. Instead of touching the wire with the finger a small piece of tinfoil may be laid between the wire and the bulb, as at A (fig. 11), and this may be touched by the finger or connected to any large object, insulated or otherwise; the effect produced is the same. Connecting the tinfoil to one of the outer coatings of the jars does not produce this effect, and it is scarcely, if at all, visible when the luminous ring is brilliant, due to a longer spark-gap. With a wire ring of several turns the author has not been able to obtain it. If a turn of bare wire be employed the effect is produced when the finger is brought very near to the wire, but if it be brought into

Fig. 11.



actual contact the effect is no longer visible. This apparently shows that it is due to the capacity between the finger or tinfoil and the wire; it is probably of the same nature as the "sensitive state" in an ordinary vacuum-tube.

Addendum, May 1st, 1893.

Since writing the above the author has made a further experiment * which at first sight appears to contradict the one † given in the paragraph on "Magnetic Effects of Discharge."

A ring (R) of four turns of wire is joined in series with a single turn, and the two are connected to the outside coatings of the jars. In the single turn a bulb is placed and the spark-gap adjusted until a fairly bright ring is produced in it at every discharge. If now a closed ring of thick copper-wire, a metal plate, or a ring of several turns, similar to R, and with its ends joined, be laid on R to act as a secondary, the luminous ring in the bulb is brighter; on substituting for this an exhausted bulb and placing it in R, there will be a brilliant ring-discharge in it, while the discharge in the other bulb will be rendered fainter or altogether extinguished. In this experiment the exhausted bulb secondary appears to act in the reverse way to a metallic secondary.

The author then made the following experiments:—

(*a*) A ring of four turns of guttapercha-covered wire precisely similar to R was made, its ends were connected to an ordinary Geissler tube. When this was used as secondary it acted exactly in the same manner as the exhausted bulb both in the first and second experiments, the Geissler tube being brilliantly illuminated.

(*b*) The Geissler tube was then removed, and the ends of the secondary coil connected to the coatings of a small Leyden jar. The effects produced by this secondary were the same as those produced by the exhausted bulb in both experiments.

* Called hereafter the second experiment. This experiment was shown when the paper was read.

† Called hereafter the first experiment.

(*c*) The ends of the secondary were connected to the loops of a glow-lamp to act as a resistance (about 100 ohms). This acted similarly to the exhausted bulb in both experiments.

(*d*) A disk of gilt paper (imitation) and also a ring of the same were used as secondaries; these acted similarly to the bulb in both experiments. When the discharge took place there were brilliant sparks produced at various spots on the paper, wherever there was any flaw in the gilding, showing that considerable energy was dissipated there.

(*e*) The secondary coil of four turns had its ends joined by a strip of gilt paper about 6 inches in length, with a considerable number of flaws in the gilding (produced purposely, by bending the paper sharply in several places, so as to obtain considerable sparking). This acted similarly to the bulb and dimmed the discharge in the bulb surrounded by the single turn. On shortening the length of gilt paper between the ends of the secondary, the discharge in the bulb was less dimmed.

The results of these five experiments are, that any of the above secondaries are able to reduce the mutual induction between the primary and secondary in the first experiment sufficiently to render faint or altogether extinguish the discharge in the bulb, and act similarly to an exhausted bulb secondary. In the second experiment a low resistance secondary behaves in the reverse manner to an exhausted bulb secondary, while (*c*) and (*e*) show that a high resistance put externally into the secondary circuit, and (*d*) that a secondary having a high resistance in itself, act in a similar manner to an exhausted bulb secondary. (*b*) shows that if the ends of the secondary be attached to a capacity it behaves like the bulb.

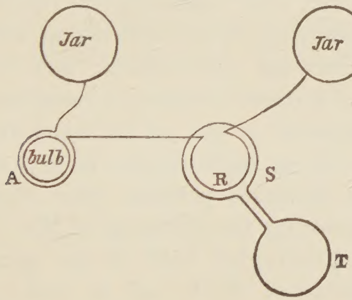
The most probable explanation seems to be the following:—The amount of energy in the jars when charged is a fixed quantity for a given spark-gap; this energy will be mostly expended in the coil R and the single turn and bulb (the second experiment). If, now, we can make energy be expended elsewhere, as in a secondary, we shall have diminished the energy received by the bulb, and this will in general dim it or altogether extinguish it. This will explain what happens when an exhausted bulb secondary is used; also experiments

(*a*), (*c*), (*d*), and (*e*). With regard to experiment (*b*), energy may have been expended in heating the glass of the jar on account of electric hysteresis. Moreover, this secondary did not dim the bulb so much as the others, but was found to be capable of improvement in this respect by including some resistance (in the shape of the glow-lamp or a strip of gilt paper) in its circuit.

In the case of a low-resistance secondary the energy dissipated in it will be small, since its impedance will not be much lessened by its being of low resistance on account of the high frequency. This does not explain, however, why the discharge in the bulb is brighter when a low-resistance secondary is used*.

A further experiment was then made. The coil R in the second experiment had a similar secondary S placed in it; this was connected to another similar coil T. The spark-gap

Fig. 12.



A one turn; S, R, and T each four turns.

was lengthened until a brilliant luminous ring was produced in a bulb placed in T. The bulb in A was then moved away from A until there was a very faint luminous ring in it. On

* This energy explanation is probably not a complete one. Working out the frequency in the cases of no secondary, a secondary of four turns short-circuited, and the same with its ends joined through 100^{ω}; the author finds that the damping-term is increased when either secondary is used, but more so with the 100^{ω} in circuit. The frequency is much the same with the 100^{ω} in circuit as when there is no secondary, but with the secondary short-circuited the frequency is about doubled. This may account for the increase in brightness of the discharge in the bulb.

removing the bulb from T a very slight brightening of the faint ring of the bulb in A was observed. Instead of placing an exhausted bulb in T, a coil of four turns with its ends joined through 100 was laid on T, and the bulb in A adjusted to give a very faint ring ; on removing the coil from T a decided brightening of the discharge in the bulb was observed. This experiment seems to show fairly conclusively that increasing the energy in the circuit of the secondary S diminishes the brightness of the discharge in the bulb placed in A*.

Discussion.

Dr. Sumpner, speaking of the apparently unclosed discharges, pointed out that they might be closed through the wire forming the primary circuit in the same way as a coil of a transformer might be arranged to act, partly as primary and partly as secondary.

Mr. A. P. Trotter, after referring to Dr. Bottomley's researches, said it was important in discussing such experiments to distinguish between electrostatic and electromagnetic effects. In Mr. Campbell Swinton's experiments the luminosity always appeared to get as far away from the wire as possible, and to be at right angles to it ; whereas in Mr. Rimington's the luminous portions were close to the wire. With a view to puzzling the discharge in Mr. Swinton's tubes, he had made a right-angled bend in the spiral surrounding the tube, the result of which was to make the luminosity discontinuous, one end of the break being bifurcated. In all Mr. Swinton's experiments brush-discharges surrounded the wire.

Prof. S. P. Thompson thought an electrostatic field would aid a discharge even if its direction was not the same as the E.M.F. due to varying magnetic induction. Planté had found that vacuum-tubes through which 800 cells were insufficient to produce a discharge, immediately allowed a discharge to pass when a rubbed ebonite rod was brought

* Since writing the above the author finds that Prof. J. J. Thomson has observed the effects noticed in the second experiment, and gives an explanation practically identical with the above.

within about ten feet distance. This effect was found to be independent of the direction of the disturbing field. Analogous effects had also been observed by Prof. Schuster, and described in his Bakerian Lecture.

Mr. E. W. Smith regarded the stresses set up in the medium as cumulative, a very slight cause acting on a substance already strained nearly to breaking-point being sufficient to cause breakdown.

Mr. Blakesley inquired if the effects were the same if the induction-coil used in one of the experiments was replaced by an electric machine, and whether the direction of the field so produced influenced the result.

Mr. W. R. Pidgeon said closed circuits were necessary ; he had found it very difficult to produce discharges in tubes unless the ends of the primary wire were brought together.

In his reply, Mr. Rimington said each turn of the luminous spiral formed a complete circuit of itself. The phenomena observed by Mr. Campbell Swinton were quite different to those he had shown and due to different causes. Mr. Swinton's spirals were reversed, and were due to phosphorescence of the glass.